Temperature equilibration rate with Fermi-Dirac statistics

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(Received 3 August 2007; revised manuscript received 24 September 2007; published 19 December 2007)

We calculate analytically the electron-ion temperature equilibration rate in a fully ionized, weakly to moderately coupled plasma, using an *exact* treatment of the Fermi-Dirac electrons. The temperature is sufficiently high so that the quantum-mechanical Born approximation to the scattering is valid. It should be emphasized that we do not build a *model* of the energy exchange mechanism, but rather, we perform a systematic first principles *calculation* of the energy exchange. At the heart of this calculation lies the method of dimensional continuation, a technique that we borrow from quantum field theory and use in a different fashion to regulate the kinetic equations in a consistent manner. We can then perform a systematic perturbation expansion and thereby obtain a finite first-principles result to leading and next-to-leading order. Unlike model building, this systematic calculation yields an estimate of its own error and thus prescribes its domain of applicability. The calculational error is small for a weakly to moderately coupled plasma, for which our result is nearly exact. It should also be emphasized that our calculation becomes unreliable for a strongly coupled plasma, where the perturbative expansion that we employ breaks down, and one must then utilize model building and computer simulations. Besides providing different and potentially useful results, we use this calculation as an opportunity to explain the method of dimensional continuation in a pedagogical fashion. Interestingly, in the regime of relevance for many inertial confinement fusion experiments, the degeneracy corrections are comparable in size to the subleading quantum correction below the Born approximation. For consistency, we therefore present this subleading quantum-to-classical transition correction in addition to the degeneracy correction.

DOI: 10.1103/PhysRevE.76.066404 PACS number(s): 52.25.Dg, 05.20.Dd, 25.45.−z, 11.10.Wx

I. INTRODUCTION

We shall calculate the thermal equilibration rate between Fermi-Dirac electrons and Maxwell-Boltzmann ions in a hot, fully ionized plasma. We shall do so *exactly* to leading and next-to-leading order in the plasma number density, and to all orders in the electron fugacity, thereby providing an essentially exact result for weakly to moderately coupled plasmas. We shall work out this problem for two reasons. First, the result is different and is needed in some applications. Second, our previous treatment in Ref. [1] of the plasma stopping power was performed in great generality, and the basic idea behind the dimensional continuation method, a somewhat subtle analytic tool that we employ, may have gotten lost in all the details. We shall use this opportunity of a simpler and specific case to treat the method in a pedagogical fashion and to explain it clearly [2].

Physical systems often contain disparate length or energy scales. For example, plasma physics involves hard collisions at short distances—ultraviolet physics, and soft interactions at large distances entailing collective effects—infrared physics. The resulting interplay of short and long distances produces the familiar Coulomb logarithm. For the electron-ion temperature equilibration rate, and for other such processes involving disparate scales, it is rather easy to calculate the leading contribution, namely, the overall factor in front of this logarithm. Although the order of magnitude of this leading order term can usually be obtained from simple dimensional analysis alone, calculating the additional dimensionless factor inside the logarithm, the subleading term, is quite difficult.

A different method [3] employing dimensional continuation has been introduced to deal with such problems, a method that makes the computation of the subleading as well as the leading contributions tractable. This method is based on tested principles of quantum field theory constructed over the last fifty years, and it has been used successfully to calculate well measured phenomena such as the Lamb shift [3], often with much more ease than traditional methods. Most recently, the method has been exploited in Ref. [1] by Brown, Preston, and Singleton (BPS) to provide an extensive treatment of the charged particle stopping power in a plasma, the energy loss per unit distance dE/dx of the charged projectile. One of the topics treated in BPS was the rate at which electrons and ions in a spatially homogeneous plasma, starting with different temperatures, come into thermal equilibrium at a common temperature. Here we shall extend this work to include the case in which the electron fugacity is sufficiently large that the electrons must be treated with a degenerate Fermi-Dirac distribution. This is the case in which Pauli blocking becomes of some importance.

The degeneracy effects that we treat here come into play as the plasma temperature is lowered. We shall calculate the rate for the general case in which the electrons are described by a Fermi-Dirac distribution, with no approximations being made to this distribution. That is, we shall perform the calculation exactly to all orders in the electron fugacity

$$z_e = e^{\beta_e \mu_e}, \tag{1.1}$$

where $\beta_e = 1/T_e$ is the reciprocal of the electron temperature and μ_e is the electron chemical potential. Note that we shall always measure temperature in energy units so that μ_e does indeed have the correct units of energy. We shall assume that the plasma is at most moderately coupled, which often implies that the degeneracy corrections are not large. Nonetheless, we shall work out the general case since this is just as

easy as treating the case of only mild degeneracy, and the general case may prove to have some application.

A plasma is seldom formed in thermal equilibrium; for example, a laser preferentially heats the light electrons rather than the heavy ions. While a nonequilibrium plasma will of course eventually thermalize, it does so in several stages. First, the electrons rapidly equilibrate among themselves to a common temperature T_e. Then, somewhat less rapidly, all the various ions equilibrate to a common temperature T_I. Finally, the electrons and ions begin the process of thermal equilibration, with the electrons delivering their energy density to the ions at a rate

$$\frac{d\mathcal{E}_{eI}}{dt} = -\mathcal{C}_{eI}(T_e - T_I). \tag{1.2}$$

The minus sign of Eq. (1.2) is a convention that renders $\mathcal{C}_{eI}$ positive when energy flows from the electrons to the ions.

In what follows, we shall calculate the rate coefficient $\mathcal{C}_{eI}$ in the limit in which the scattering contribution is dominated by quantum diffraction. We first establish our conventions and notation in Sec. III, while in Sec. IV we describe the calculational method that we employ in a detailed, pedagogical fashion since it involves a different and subtle technique. The relation of our method to older approaches is sketched in Sec. IV, and for clarity, some simple thermodynamical relations are reviewed in Sec. VI. Our method combines in an unambiguous way the short-distance physics given by a Boltzmann equation that includes the Pauli blocking of the electrons and the long-distance physics described by a Lenard-Balescu equation modified to include the effects of this Pauli blocking. The explicit computation in these two regimes is presented in Secs. VII and VIII. These results are combined in Sec. IX to obtain the exact result for the rate coefficient $\mathcal{C}_{eI}$ to leading and next-to-leading order in the plasma density and to all orders in the electron fugacity z_e. Appendix A contains the details of the plasma dielectric function needed in the text.

The effects of Fermi-Dirac statistics and the accompanying Pauli blocking only become appreciable as the temperature is lowered below the limit in which the scattering is entirely dominated by quantum diffraction. Along with the degeneracy correction, we must therefore include the first classical correction, since these two mechanisms may become of the same order of magnitude. Appendix B extracts the first classical correction from the general result given in BPS [1].

As this outline of the paper shows, we must perform a considerable amount of work to calculate $\mathcal{C}_{eI}$ in an analytic form. Hence to motivate our development, and to provide a summary for a reader who may not be interested in all the mathematical details, we first turn in Sec. II to provide a brief compendium of our results. We do, however, urge the reader to work through the main body of the text, since there is no substitute for the calculation itself in illustrating the relevant physics.

II. COMPENDIUM OF RESULTS

To provide an overview of our results, we start with the result (9.9) for the rate coefficient, which we display below for convenience. This expression for $\mathcal{C}_{eI}$ is exact to all orders in the electron fugacity, but valid only in a temperature regime in which the short-distance scattering is described by the quantum-mechanical first Born approximation:

$$\mathcal{C}_{eI} = \frac{\omega_I^2}{2\pi}\sqrt{\frac{\beta_e m_e}{2\pi}}\left(\beta_e e^2 \frac{2e^{\beta_e\mu_e}}{\lambda_e^3}\right)\left\{\frac{\ln\Lambda}{\exp\{\beta_e\mu_e\}+1} + \frac{1}{2}\sum_{l=1}^{\infty}(-1)^{l+1}\ln\{l+1\}e^{l\beta_e\mu_e}\right\}, \tag{2.1}$$

where

$$\ln\Lambda = \frac{1}{2}\left[\ln\left\{\frac{16\pi}{\kappa_e^2\lambda_e^2}\right\} - \gamma - 1\right]. \tag{2.2}$$

Here ω_I^2 is the sum of the squared ion plasma frequencies ω_i^2, m_e is the electron mass, λ_e is the electron thermal wavelength, and κ_e^2 is the electron contribution to the squared Debye wave number, *including* electron degeneracy effects. The precise definition of these quantities is presented in Sec. III, but we should note here that we employ rationalized Gaussian units, so that the energy of two electrons of charge e a distance r apart is given by $e^2/4\pi r$. The structure of the first line that appears in our result (2.1) agrees with the previous result of Brysk [4] when his Eq. (35) is re-expressed in terms of our notation. However, Brysk does not obtain the precise result (2.2) for $\ln\Lambda$, but rather only an approximate, leading-log evaluation of the usual form $\ln\{b_{\max}/b_{\min}\}$. Moreover, Brysk [4] also does not obtain the second line of our result (2.1). This second line does not contribute in the Maxwell-Boltzmann limit of very small fugactiy z_e, but it does a provide a significant first-order correction in z_e.

The previous work of BPS [1] computed the exact temperature equilibration rate for a weakly to moderately coupled nondegenerate plasma. While this general result was rather complicated, the high-temperature limit, in which the short-distance Coulomb scattering is given by its first Born approximation, has a rather simple form. The nondegenerate rate coefficient of BPS can be obtained from Eq. (2.1) by taking the small-fugacity limit $z_e \to 0$, which gives

$$\mathcal{C}_{eI}^{\text{non-degen}} = \frac{\omega_I^2}{2\pi}\sqrt{\frac{\beta_e m_e}{2\pi}}(\beta_e e^2 n_e)\ln\Lambda_0, \tag{2.3}$$

where the logarithm $\ln\Lambda_0$ above is the nondegenerate limit of Eq. (2.2). The term

$$\kappa_{e0}^2 \equiv \beta_e e^2 n_e \tag{2.4}$$

in parentheses follows from the well known relation (3.12) between number density and fugacity, and it is just the square of the nondegenerate form of the electron's Debye wave number. The nondegenerate limit of Eq. (2.2) is accomplished by the substitution $\kappa_e \to \kappa_{e0}$, and expressing this result in terms of the electron plasma frequency provides the form

$$\ln \Lambda_0 = \frac{1}{2}\left[\ln\left\{ \frac{8T_e^2}{\hbar^2\omega_e^2} \right\} - \gamma - 1 \right], \tag{2.5}$$

where we have used the relation $\kappa_{e0}^2\lambda_e^2/2\pi = \hbar^2\omega_e^2/T_e^2$. The rate coefficient of Eqs. (2.3) and (2.5) is just that given previously by Eq. (12.12) of BPS [1], an expression that was also quoted in Eq. (3.61) in the introductory portion of that work. There is, however, one difference between Eq. (2.5) and the result (12.12) of BPS. Namely, the correct term $-\gamma - 1$ appearing in Eq. (2.5) above was incorrectly written as $-\gamma - 2$ in Eqs. (12.12) and (3.61) of BPS because of a transcription error in passing between Eqs. (12.43) and (12.44) of BPS. We have taken the opportunity here to correct this mistake.

The general structure of the "Coulomb log" $\ln \Lambda_0$ with its dependence upon the temperature has, of course, long been known [5]. Equation (2.5), however, gives the precise definition of the Coulomb log (in the nondegenerate limit) for the temperature equilibration process we have been discussing, including the correct additional constant terms, namely, the terms $\ln 8 - \gamma - 1$. We should, however, emphasize two points. First of all, the Coulomb log is by no means a universal quantity, but rather its precise form is process dependent. For example, the Coulomb log for electron conductivity differs from the Coulomb log for the electron-ion thermal relaxation rate given here [6]. Second, we should emphasize that the Coulomb log for the relaxation rate does not depend upon the ion temperature T_I but only upon the electron temperature T_e. Some authors [6] incorrectly replace the squared electron Debye wave number κ_e^2 by the fully screened Debye wave number $\kappa_D^2 = \kappa_e^2 + \kappa_I^2$. This incorrectly includes the ion contribution κ_I^2, and thus introduces a spurious dependence on the ion temperature T_I.

If the plasma temperature becomes very low, then the population of bound states must be taken into account. In such a regime, our assumption that the plasma is fully ionized and weakly coupled breaks down, and our calculation is no longer reliable. Hence we shall discuss and examine in detail only the limit in which the degeneracy corrections are mild, the regime in which the temperature is not too low and the electron fugacity is not too large. In this limit, only the first-order fugacity contribution from the general result (2.1) need be retained, and therefore the second line in Eq. (2.1), the term omitted by Brysk [4], makes an essential contribution. As the temperature is lowered, however, the subleading contribution to the Born approximation becomes comparable to the degeneracy correction, and it too must also be accounted for. In Appendix B, we therefore extract this subleading correction in the transition region between quantum and classical scattering from the general result given in BPS. In the following, these two types of corrections will be called the degeneracy correction and the (first) quantum-to-classical transition correction.

Working to leading order in the fugacity, we only need to expand Eq. (2.1) to linear order in $z_e = e^{\beta_e \mu_e}$, a result contained in Eq. (9.15). The quantum-to-classical transition correction is contained in Eq. (B33). Expressing the fugacity correction in terms of the density according to Eq. (3.18), the rate coefficient $\mathcal{C}_{eI}$ reads

$$\begin{aligned}\mathcal{C}_{eI} \simeq \frac{\omega_I^2}{2\pi}\sqrt{\frac{\beta_e m_e}{2\pi}}(\beta_e e^2 n_e)\Bigg[&\frac{1}{2}\left[\ln\left\{ \frac{8T_e^2}{\hbar^2\omega_e^2} \right\} - \gamma - 1 \right] \\ &+ \frac{n_e\lambda_e^3}{2}\Bigg\{ -\left(1 - \frac{1}{2^{3/2}}\right)\frac{1}{2}\left[\ln\left\{ \frac{8T_e^2}{\hbar^2\omega_e^2} \right\} - \gamma - 1 \right] \\ &+ \left(\frac{1}{2}\ln 2 + \frac{1}{2^{5/2}} \right)\Bigg\} - \frac{\epsilon_H}{T_e}\sum_i \frac{Z_i^2\omega_i^2}{\omega_I^2} \\ &\times\left[\zeta(3)\left(\ln\left\{ \frac{T_e}{Z_i^2\epsilon_H} \right\} - \gamma \right) - 2\zeta'(3) \right]\Bigg],\end{aligned} \tag{2.6}$$

where the numerical values of the zeta function and its derivative are

$$\zeta(3) = \sum_{k=1}^{\infty}\frac{1}{k^3} = 1.202\,05\ldots, \tag{2.7}$$

and

$$\zeta'(3) = -\sum_{k=1}^{\infty}\frac{1}{k^3}\ln k = -0.198\,12\ldots. \tag{2.8}$$

We also write Z_i as the ionic charges in units of the electron charge e. Before examining expression (2.6) in detail, we note that the first line is the leading rate coefficient calculated in BPS [1], Eq. (2.3) above; the second term in curly braces is the first degeneracy correction following from Eq. (2.1); and the remaining third term corresponds to the first quantum-to-classical transition calculated in Appendix B.

In the last term of Eq. (2.6), the ratio ϵ_H/T_e describes the relative size of the correction, where

$$\epsilon_H = \left(\frac{e^2}{4\pi}\right)^2\frac{m_e}{2\hbar^2} \simeq 13.6\ \text{eV} \tag{2.9}$$

is the binding energy of the hydrogen atom. For some temperature and number density regimes of interest, the second and third terms in Eq. (2.6) become comparable in size. Hence while our main thrust in this paper is degeneracy corrections, we must also take into account this first quantum-to-classical transition.

It is conventional to write the Coulomb logarithm as $\ln\{b_{\max}/b_{\min}\}$, where $b_{\max}$ is a Debye length long-distance cutoff, while $b_{\min}$ is a short-distance cutoff that, depending upon the circumstances, is either a classical distance of closest approach $b_{\text{cl}} \sim e^2/T_e$ or a quantum wave length $b_{\text{qm}} \sim \lambda_e$. Often, an interpolation is made [7] between these two limits by writing

$$b_{\min}^2 = b_{\text{cl}}^2 + b_{\text{qm}}^2 = b_{\text{qm}}^2\left[1 + \frac{b_{\text{cl}}^2}{b_{\text{qm}}^2} \right] \sim b_{\text{qm}}^2\left[1 + \frac{\epsilon_H}{T_e} \right]. \tag{2.10}$$

Such an interpolation gives a first correction proportional to the proper quantum expansion parameter ϵ_H/T_e, but it fails entirely to produce the proper logarithmic behavior $(\epsilon_H/T_e)\ln\{\epsilon_H/T_e\}$ displayed in the last term of Eq. (2.6).

Figures 1 and 2 illustrate the size of the degeneracy and the first quantum-to-classical transition corrections in an

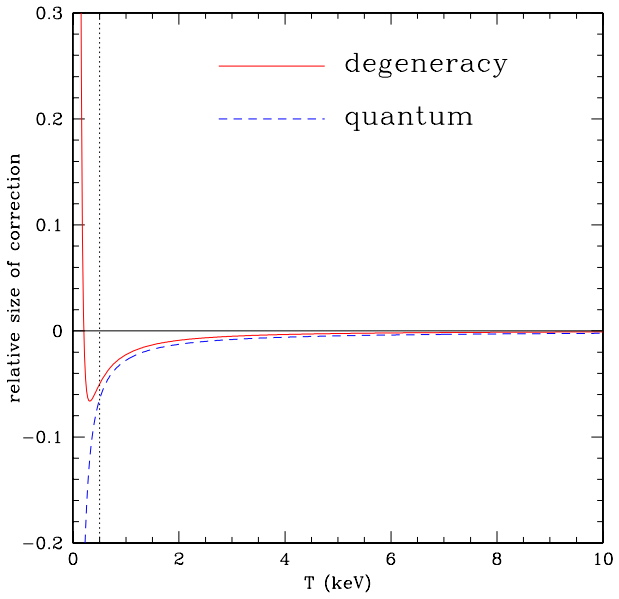

FIG. 1. (Color online) The relative size of the degeneracy correction and the first classical-to-quantum correction as a function of temperature in keV. The plasma is equimolar deuterium-tritium at an electron number density $n_e=10^{25}$ cm^{-3}. These corrections are relative to the leading nondegenerate BPS rate: the degeneracy correction (the solid line) is the ratio of the second to the first line in Eq. (2.1), while the classical-to-quantum correction (the dashed line) is the ratio of the third to the first. The electron temperature runs between values 0.1 and 10 keV. Our calculation ceases to be valid at low temperatures, and this is indicated by the vertical dotted line.

equimolar deuterium-tritium plasma, for electron number densities of $n_e=10^{25}$ cm^{-3} and $n_e=10^{26}$ cm^{-3}, respectively. The solid curves denote the size of the degeneracy corrections relative to the leading BPS term, the ratio of the second

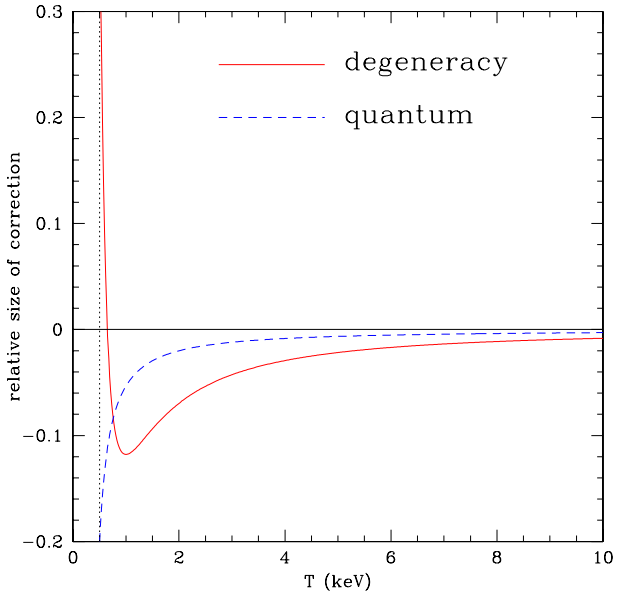

FIG. 2. (Color online) Same as Fig. 1, except the number density is $n_e=10^{26}$ cm^{-3}. Again, the solid line is the degeneracy correction and the dashed line is the first quantum-to-classical correction.

to the first term of Eq. (2.6). The dashed curves show the corresponding size of the first quantum-to-classical transition, the ratio of the third to the first term of Eq. (2.6). Note that both corrections are comparable in magnitude between these two electron number densities. We have plotted the corrections for temperatures between $T_e=0.1$ keV and $T_e=10$ keV. However, below about 0.5 keV, ionization and other strongly coupled plasma effects become important, and our formalism breaks down. Also, at these lower temperatures, higher order fugacity terms in Eq. (2.1) become important, and this could change the low-temperature behavior of the solid curves; therefore, one should trust Eq. (2.6) only to the right of the vertical dotted line at $T_e=0.5$ keV. For densities below $n_e=10^{25}$ cm^{-3}, the degeneracy correction is much smaller than the quantum-to-classical correction. The situation is reversed for densities greater than $n_e=10^{26}$ cm^{-3}, where degeneracy effects dominate over the quantum-to-classical corrections. Interestingly, the density and temperature range in which the degeneracy and the quantum-to-classical corrections are comparable lies in the regime relevant for inertial confinement fusion.

III. CONVENTIONS AND NOTATION

We will treat the ions with Maxwell-Boltzmann statistics and the electrons with Fermi-Dirac statistics. The thermal equilibrium form of an ion phase space density $f_i(\mathbf{p}_i)$ thus reads

$$f_i(\mathbf{p}_i) = e^{-\beta_i[E_i(\mathbf{p}_i)-\mu_i]}, \tag{3.1}$$

while for electrons in thermal equilibrium,

$$f_e(\mathbf{p}_e) = \frac{1}{e^{\beta_e[E_e(\mathbf{p}_e)-\mu_e]}+1}. \tag{3.2}$$

Letting the index b refer to either the ions i or the electrons e, the kinetic energy is

$$E_b(\mathbf{p}_b) = \frac{p_b^2}{2m_b}, \tag{3.3}$$

and the inverse temperature and chemical potential are

$$\beta_b = 1/T_b \quad \text{and} \quad \mu_b. \tag{3.4}$$

Since, in our intermediate calculations, we work in an arbitrary number of dimensions ν, each species number density will appear as

$$n_b = \mathfrak{g}_b \int \frac{d^\nu p_b}{(2\pi\hbar)^\nu} f_b(\mathbf{p}_b), \tag{3.5}$$

where $\mathfrak{g}_b$ is spin-degeneracy factor. For electrons $\mathfrak{g}_e=2$. We are using a notation for the distribution functions f_b in which the species index b implicitly includes spin degrees of freedom. It is inconvenient, however, to use this notation for the number density n_b. Measurements of the species density are usually insensitive to spin degrees of freedom, and we shall therefore denote the number density of the species (including all the spins) by n_b. This accounts for the factor of $\mathfrak{g}_b$ in Eq. (3.5). For the ions in thermal equilibrium, the integral (3.5) is

a product of trivial Gaussian integrals, and so

$$n_i = \mathfrak{g}_i \lambda_i^{-\nu} e^{\beta_i \mu_i}, \tag{3.6}$$

where we define the thermal wavelength for species b as

$$\lambda_b = \hbar \left(\frac{2\pi\beta_b}{m_b} \right)^{1/2}. \tag{3.7}$$

For the electrons in thermal equilibrium, we first pass to hyperspherical coordinates to write

$$n_e = \mathfrak{g}_e \frac{\Omega_{\nu-1}}{(2\pi\hbar)^\nu} \int_0^\infty p^{\nu-1} dp \frac{1}{e^{\beta_e[E_e(\mathbf{p})-\mu_e]}+1}. \tag{3.8}$$

Here $\Omega_{\nu-1}$ is the area of a unit hypersphere in a space of ν dimensions that is evaluated in the next section with the result

$$\Omega_{\nu-1} = \frac{2\pi^{\nu/2}}{\Gamma(\nu/2)}. \tag{3.9}$$

Changing to dimensionless variables,

$$x = \beta_e E_e(\mathbf{p}) = \beta_e \frac{p^2}{2m_e}, \tag{3.10}$$

allows us to express the electron number density as

$$n_e = \mathfrak{g}_e \lambda_e^{-\nu} \frac{1}{\Gamma(\nu/2)} \int_0^\infty \frac{dx}{x} \frac{x^{\nu/2}}{e^{-\beta_e\mu_e} e^x + 1}. \tag{3.11}$$

When the quantity $-\beta_e\mu_e$ becomes very large (and positive), Fermi-Dirac statistics pass to the Maxwell-Boltzmann limit, with the denominator above becoming a simple exponential. In this limit, the x integration becomes the standard representation of $\Gamma(\nu/2)$, and we see that the number density in this small fugacity limit is given by the Maxwell-Boltzmann form (3.6), as it must be. Expanding Eq. (3.11) to second order in the fugacity $z_e = e^{\beta_e\mu_e}$ and using $\mathfrak{g}_e = 2$ gives, for the the physical case of three dimensions,

$$\nu = 3: \quad n_e \simeq \frac{2}{\lambda_e^3} e^{\beta_e\mu_e} \left[1 - \frac{e^{\beta_e\mu_e}}{2^{3/2}} \right]. \tag{3.12}$$

Note that the first correction, which decreases the number density, is simply the fugacity divided by a numerical factor of order unity. By small fugacity, we therefore mean that $e^{\beta_e\mu_e} \ll 2^{3/2} \simeq 2.8$.

As we shall see in the following section, in a space of ν dimensions, the energy of two charges e a distance r apart is proportional to $e^2/r^{\nu-2}$. Since the units of a number density n are $(\text{length})^{-\nu}$, we conclude that $e^2 n$ has the units of energy over length squared, independently of the spatial dimension ν. In particular,

$$\omega_b^2 = \frac{e_b^2 n_b}{m_b} \tag{3.13}$$

is the squared plasma frequency for species b with the fixed dimension of an inverse-time squared, regardless of the spatial dimensionality ν. The situation for the squared Debye wave number is essentially the same, except that, as noted in Appendix A, in general, this quantity is defined in terms of the fluctuations in the number density, and so

$$\kappa_b^2 = \beta_b e_b^2 \frac{\partial n_b}{\partial(\beta_b\mu_b)}. \tag{3.14}$$

For Maxwell-Boltzmann statistics, the derivative that appears here just reproduces the particle density in accord with the fact that classical particles are described by Poisson statistics. However, for Fermi-Dirac statistics, one must use

$$\frac{\partial n_b}{\partial(\beta_b\mu_b)} = \frac{\mathfrak{g}_b \lambda_b^{-\nu}}{\Gamma(\nu/2)} \int_0^\infty \frac{dx}{x} \frac{x^{\nu/2} e^{-\beta_b\mu_b} e^x}{[e^{-\beta_b\mu_b} e^x + 1]^2} \leqslant n_b. \tag{3.15}$$

The inequality that appears here implies that

$$\kappa_b^2 \leqslant \beta_b e_b^2 n_b = \kappa_{b0}^2. \tag{3.16}$$

For the dilute case in three dimensions, including the first correction in the fugacity, the electron Debye wave number is given by

$$\begin{aligned} \nu = 3: \quad \kappa_e^2 &\simeq \beta_e e^2 \frac{2}{\lambda_e^3} e^{\beta_e\mu_e} \left[1 - \frac{2}{2^{3/2}} e^{\beta_e\mu_e} \right] \\ &\simeq \beta_e e^2 n_e \left[1 - \frac{1}{2^{3/2}} e^{\beta_e\mu_e} \right]. \end{aligned} \tag{3.17}$$

In the first-order fugacity correction that appears here we can use the lowest-order result

$$n_e \simeq \frac{2}{\lambda_e^3} e^{\beta_e\mu_e} \tag{3.18}$$

to compute the fugacity and thus write

$$\nu = 3: \quad \kappa_e^2 \simeq \beta_e e^2 n_e \left[1 - \frac{1}{2^{3/2}} \frac{\lambda_e^3 n_e}{2} \right]. \tag{3.19}$$

IV. METHOD

Since the method of dimensional continuation that we shall use is different and perhaps subtle, we present here a pedagogical account of its basis.

A. Disparate length scales; expansion parameter

The electron-ion energy exchange brought about by their collisions in a plasma involves a Coulomb interaction that is Debye screened at large distances and, as we shall see in the course of our work, cut off at short distances by quantum effects. As we shall sketch below in Sec. IV C for arbitrary spatial dimensions ν, the familiar elementary description of this energy transfer for $\nu = 3$ dimensions involves the impact parameter integral

$$\int_{b_{\min}}^{b_{\max}} \frac{db}{b} = \ln\left\{ \frac{b_{\max}}{b_{\min}} \right\}. \tag{4.1}$$

Here $b_{\min}$ is the minimum distance of closest approach that, in the quantum limit that is relevant here, is set by the scale of the electron thermal wavelength λ_e. That is, $b_{\min}$ is some numerical constant times λ_e. The upper limit on the impact

parameter integral is set by the electron Debye length κ_e^{-1}, with $b_{\max}$ some numerical multiple of κ_e^{-1}. Thus

$$\frac{b_{\min}}{b_{\max}} \sim \lambda_e \kappa_e. \tag{4.2}$$

The purpose of the dimensional continuation method is to precisely determine the numerical constants that appear here.

Our method applies when the ratio $b_{\max}/b_{\min}$ is large: In this case, the dimensionless parameter $\lambda_e \kappa_e$ is small, and we shall use it as our expansion parameter. As we noted in the previous section, the Debye wave number κ_e always has the dimensions of an inverse length, even at arbitrary spatial dimension ν. Hence, $\lambda_e \kappa_e$ is a convenient parameter to employ in our dimensional continuation scheme because it remains dimensionless as the number of spatial dimensions ν is varied. Moreover, as we shall see, it is the combination $\lambda_e \kappa_e$ that directly arises as our computations progress.

The electron plasma coupling strength is characterized by the ratio of the Coulomb electrostatic energy of two electrons a Debye distance apart divided by the temperature. In the physical space of three dimensions, $\nu=3$, this is the dimensionless parameter

$$\nu = 3: \quad g_e = \beta_e \frac{e^2 \kappa_e}{4\pi}. \tag{4.3}$$

The perturbative expansion of plasma thermodynamic parameters involve a series of ascending *integer* powers (up to additional logarithmic corrections) of the coupling constant g_e. Except for different conventions that can alter a trivial overall factor, the electron quantum Coulomb parameter is defined as the Coulomb energy for two electrons a thermal wavelength apart divided by the temperature. For three dimensions, this reads

$$\nu = 3: \quad \eta_e = \beta_e \frac{e^2}{4\pi \lambda_e}. \tag{4.4}$$

Hence, in three dimensions,

$$\nu = 3: \quad \lambda_e \kappa_e = \frac{g_e}{\eta_e}. \tag{4.5}$$

Thus our expansion parameter $\lambda_e \kappa_e$ is essentially the plasma coupling parameter g_e, albeit divided by the quantum parameter η_e. Accordingly, one could equivalently work in terms of the coupling g_e as we have done in the past [1], but here it is more convenient to use $\lambda_e \kappa_e$, and so this we shall do.

Our work applies to fully ionized plasmas where the temperature is large and thus the parameter η_e is small. The condition that $\lambda_e \kappa_e$ be small requires that the plasma coupling g_e be even smaller than η_e. To put this in perspective, we recall that even if Fermi-Dirac statistics are required, the Debye wave number is smaller than that given by the Maxwell-Boltzmann form with the same temperature and density. Hence we have

$$\lambda_e^2 \kappa_e^2 \leqslant \lambda_e^2 \beta_e e^2 n_e, \tag{4.6}$$

where the electron number density on the right-hand side of this equation is given by Maxwell-Boltzmann statistics. Using now the number density (3.6) in the Maxwell-Boltzmann limit and the definition (3.7) of the thermal wavelength, we find that

$$\lambda_e^2 \beta_e e^2 n_e = 8\pi^{1/2} e^{\beta_e \mu_e} \sqrt{\frac{\epsilon_H}{T_e}} = 8\pi^{1/2} e^{\beta_e \mu_e} \sqrt{\frac{13.6 \text{ eV}}{T_e}}. \tag{4.7}$$

Thus even for somewhat large electron fugacities $e^{\beta_e \mu_e}$, the expansion parameter $\lambda_e \kappa_e$ will be small provided the temperature is reasonably large.

To explain further the utility of $\lambda_e \kappa_e$ as the appropriate expansion parameter, we examine the situation when the spatial dimension ν departs from its physical value $\nu=3$. In this case, as we shall soon see in the next subsection, the Coulomb potential a distance r away from a point charge has the dependence $r^{-(\nu-2)}$. Thus the plasma coupling and quantum Coulomb parameters have the form

$$g_e \sim \beta_e e^2 \kappa_e^{\nu-2}, \quad \eta_e \sim \beta_e \frac{e^2}{\lambda_e^{\nu-2}}, \tag{4.8}$$

and so $g_e/\eta_e \sim (\lambda_e \kappa_e)^{\nu-2}$ or

$$\lambda_e \kappa_e \sim \left(\frac{g_e}{\eta_e}\right)^{1/(\nu-2)}. \tag{4.9}$$

This emphasizes that although the form of the coupling $\lambda_e \kappa_e$ that we employ here does not change as the spatial dimension is altered, its form in terms of g_e/η_e does depend upon this dimensionality.

B. Idea of dimensional continuation

We have already seen explicitly how a geometrical quantity, namely, the number density, can be computed in a space of arbitrary dimensionality ν. In fact, all fundamental theories can be formulated in a world that has space of arbitrary dimensionality. Modern quantum field theory, the mother of all physical theory, is generally formulated for spaces of arbitrary dimensionality in order to regulate it. (See, for example, Ref. [9].) The well known Born-Bogoliubov-Green-Kirkwood-Yvon (BBGKY) hierarchy of coupled equations that depicts general kinetics can obviously be written in a space of arbitrary dimensionality ν. For $\nu>3$, the Coulomb force acts as a short-range force; for $\nu<3$, it acts a long-range force. Although the complete BBGKY set of coupled equations for Coulomb forces must remain valid for arbitrary spatial dimensionality ν, it cannot be approximated by the Boltzmann or Lenard-Balescu equations for general ν values. To leading order in the density, the Boltzmann equation describes the short-distance, hard scattering correctly while the Lenard-Balescu equation correctly describes the long-distance, collective interactions. Hence, to this leading order, the BBGKY hierarchy of equations reduces to the Boltzmann equation for $\nu>3$, but for $\nu<3$, the BBGKY hierarchy reduces to the Lenard-Balescu equation. We shall see how this works out in detail as our work progresses.

Here we introduce the idea of dimensional continuation by examining the case of electrostatics. The Poisson equation for a point charge Ze in ν dimensions reads

$$-\nabla^2\phi^{(\nu)}(\mathbf{r}) = Ze\,\delta^{(\nu)}(\mathbf{r}). \tag{4.10}$$

Its solution may be expressed as a Fourier integral:

$$\phi^{(\nu)}(\mathbf{r}) = \int \frac{d^\nu k}{(2\pi)^\nu}\frac{Ze}{\mathbf{k}^2}e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{4.11}$$

As it stands, this integral is defined for all positive integer dimensions ν. As is well known from the theory of complex functions, an analytic function is defined from its values on the positive real integers, provided that the function does not diverge rapidly at infinity in the complex plane [8]. But for our equations, we can obtain this extension by explicit calculations. For the case at hand, we first write

$$\frac{1}{\mathbf{k}^2} \equiv \frac{1}{k^2} = \int_0^\infty ds e^{-k^2 s}, \tag{4.12}$$

and interchange integrals to encounter

$$\begin{aligned}\int \frac{d^\nu k}{(2\pi)^\nu}e^{-k^2 s}e^{i\mathbf{k}\cdot\mathbf{r}} &= \int \frac{d^\nu k}{(2\pi)^\nu}\exp\left\{-\left[\mathbf{k} - i\frac{\mathbf{r}}{2s}\right]^2 s\right\} \\ &\quad\times\exp\left\{-\frac{r^2}{4s}\right\} \\ &= \left(\frac{1}{4\pi s}\right)^{\nu/2}\exp\left\{-\frac{r^2}{4s}\right\}.\end{aligned} \tag{4.13}$$

Here we have completed the square and used the variable in the square brackets as the new integration variable to obtain a product of ν ordinary Gaussian integrals whose evaluation produces the final result. The change of variables from s to $t = r^2/4s$ now gives

$$\begin{aligned}\phi^{(\nu)}(\mathbf{r}) &= \frac{Ze}{r^{\nu-2}}\left(\frac{1}{\pi}\right)^{\nu/2}\frac{1}{4}\int_0^\infty \frac{dt}{t}t^{(\nu-2)/2}e^{-t} \\ &= \frac{Ze}{r^{\nu-2}}\left(\frac{1}{\pi}\right)^{\nu/2}\frac{1}{4}\Gamma\left(\frac{\nu-2}{2}\right),\end{aligned} \tag{4.14}$$

since the t integral is a standard representation of the gamma function. This result now defines an electrostatic potential for any value of ν in the entire complex plane.

As a mathematical application of this result, we note that it gives the electric field

$$\mathbf{E}(\mathbf{r}) = \frac{Ze\hat{\mathbf{r}}}{r^{\nu-1}}\left(\frac{1}{\pi}\right)^{\nu/2}\frac{1}{2}\Gamma\left(\frac{\nu}{2}\right). \tag{4.15}$$

Hence Gauss' law applied to a sphere of radius r,

$$S(r)\hat{\mathbf{r}}\cdot\mathbf{E}(\mathbf{r}) = Ze, \tag{4.16}$$

informs us that this sphere has an area $S = \Omega_{\nu-1}r^{\nu-1}$ where

$$\Omega_{\nu-1} = \frac{2\pi^{\nu/2}}{\Gamma(\nu/2)} \tag{4.17}$$

is the area of a $(\nu-1)$-dimensional unit hypersphere embedded in the ν-dimensional space.

There are physical implications that follow from Eq. (4.14) of the electrostatic potential of a point charge in ν dimensions. These are brought out in Fig. 3. As the figure shows, the Coulomb potential of a point charge becomes more singular at the origin as the spatial dimension ν increases: the physics at short distances is increasingly emphasized as the spatial dimension ν increases. Since short distances correspond to high wave numbers, this is equivalent to stating that large ν emphasizes ultraviolet physical processes. Conversely, as the spatial dimension ν decreases, the potential falls off less rapidly at large distances: the physics at large distances becomes ever more important as the spatial dimension ν decreases. Since long distances correspond to low wave numbers, this is equivalent to stating that small ν emphasizes infrared physical processes. As we shall see, the electron-ion energy exchange can be computed with the Boltzmann equation for $\nu > 3$ since it correctly accounts for hard scattering. The result, however, has a simple pole that diverges as $\nu \to 3$ from above. Conversely, the electron-ion energy exchange can be computed from the Lenard-Balescu equation for $\nu < 3$ since it correctly accounts for the long-range screening brought about by the collective, dielectric effects in the plasma. The result, however, has a simple pole and diverges as $\nu \to 3$ from below. These general features are brought out in the simple computation of the next subsection.

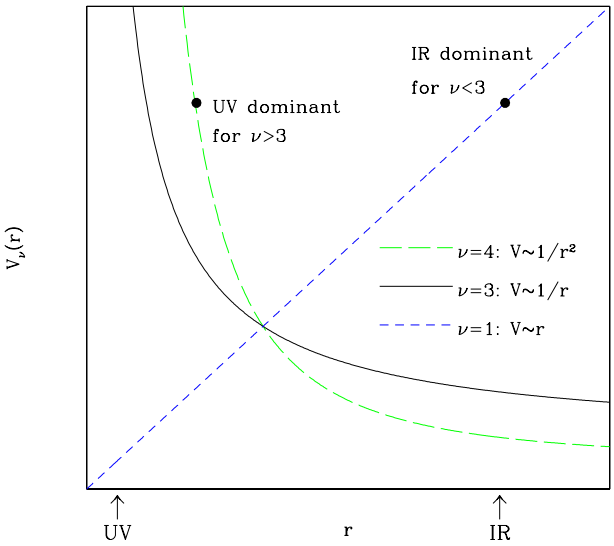

FIG. 3. (Color online) Short-distance or ultraviolet (UV) physics dominates in dimensions $\nu > 3$. Long-distance or infrared (IR) physics dominates when $\nu < 3$. UV and IR physics are equally important in $\nu = 3$.

C. Energy loss structure in ν spatial dimensions

To illustrate the remarks that we have been making, we consider the lowest-order energy loss of an electron passing by a fixed point of charge Ze. In zeroth order, an electron with impact parameter $\mathbf{b}$ and perpendicular velocity $\mathbf{v}$, so that $\mathbf{b}\cdot\mathbf{v} = 0$, simply follows the straight line $\mathbf{b} + \mathbf{v}t$ as a function of the time t. In first approximation, with the electric field $\mathbf{E}$ given by Eq. (4.15), the electron acquires a momentum transfer

$$\Delta\mathbf{p} = -e\int_{-\infty}^{+\infty} dt\mathbf{E}(\mathbf{b} + \mathbf{v}t) \tag{4.18}$$

in passing by the fixed point charge Ze. This gives an energy change

$$\Delta E = \frac{\Delta \mathbf{p}^2}{2m_e}. \tag{4.19}$$

It is a straightforward matter to check that this gives, up to numerical factors of no importance,

$$\Delta E \sim \frac{1}{m_e}\left(\frac{Ze^2}{v}\right)^2\left(\frac{1}{b^{\nu-2}}\right)^2. \tag{4.20}$$

In ν spatial dimensions, an element of cross section is given by $d\sigma = \Omega_{\nu-2} b^{\nu-2} db$. Hence the weighted energy loss has the form

$$\int d\sigma \Delta E \sim \int_{b_{\min}}^{b_{\max}} \frac{db}{b^{\nu-2}}. \tag{4.21}$$

This example explicitly demonstrates that large ν is dominated by short-distance physics and small ν is dominated by long-distance physics. Moreover, it shows explicitly that $\nu = 3$ is the dividing line between these two regions. To bring this out, all we need do is to note that for $\nu > 3$ the impact parameter integral is not sensitive to the large distance cutoff, and we may take the limit $b_{\max} \to \infty$ to obtain

$$\nu > 3: \quad I^{>}(\nu) = \int_{b_{\min}}^{\infty} \frac{db}{b^{\nu-2}} = \frac{b_{\min}^{3-\nu}}{\nu - 3}. \tag{4.22}$$

Conversely, for $\nu < 3$, we may set $b_{\min} = 0$, with

$$\nu < 3: \quad I^{<}(\nu) = \int_{0}^{b_{\max}} \frac{db}{b^{\nu-2}} = \frac{b_{\max}^{3-\nu}}{3 - \nu}. \tag{4.23}$$

The results displayed are the dominant forms in the two different regions of spatial dimensionality ν.

D. Implementation of dimensional continuation

The situation that we have just described leads to a well defined result because it is akin to the following example. Suppose that we have a theory that is well defined in the neighborhood of the physical dimension $\nu = 3$, and that the theory contains a small parameter ϵ. Moreover, suppose that we need to evaluate a function F that depends upon this small parameter ϵ in the following fashion. For $\nu > 3$ the leading behavior of F goes like $\epsilon^{3-\nu}$, and the function F has a simple pole in ν as $\nu \to 3$ from above. Conversely, for $\nu < 3$, the leading behavior of the function F goes like $\epsilon^{\nu-3}$ and the function F has a simple pole in ν as $\nu \to 3$ from below. That is, we have the leading terms

$$\nu > 3: \quad F^{>}(\nu;\epsilon) = A^{>}(\nu)\epsilon^{3-\nu}, \tag{4.24}$$

and

$$\nu < 3: \quad F^{<}(\nu;\epsilon) = A^{<}(\nu)\epsilon^{\nu-3}. \tag{4.25}$$

Since the two contributions each have poles in ν,

$$A^{>}(\nu) = \frac{R^{>}}{\nu - 3} + r^{>} + \mathcal{O}(\nu - 3), \tag{4.26}$$

and

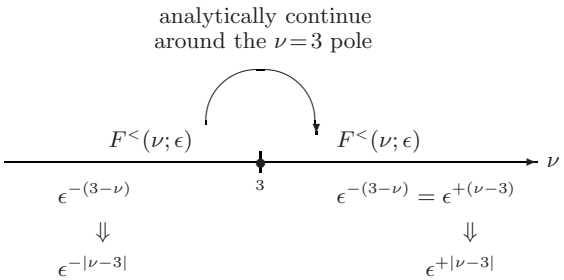

FIG. 4. The analytic continuation of $F^{<}(\nu;\epsilon)$ from $\nu < 3$ to the region $\nu > 3$: the same expression can be used for $F^{<}(\nu;\epsilon)$ throughout the complex plane since the pole at $\nu = 3$ can easily be avoided. Note that the quantity $F^{<}(\nu;\epsilon) \sim \epsilon^{(\nu-3)}$ is leading order in ϵ for $\nu < 3$. However, upon analytically continuing to $\nu > 3$ we find that $F^{<}(\nu;\epsilon) \sim \epsilon^{|\nu-3|}$ which is next-to-leading order in ϵ relative to $F^{>}(\nu;\epsilon) \sim \epsilon^{-|\nu-3|}$.

$$A^{<}(\nu) = \frac{R^{<}}{3 - \nu} + r^{<} + \mathcal{O}(3 - \nu). \tag{4.27}$$

The function $F^{<}(\nu;\epsilon)$ is of leading order in the expansion parameter ϵ for $\nu < 3$. Since it is an analytic function of ν, it may be continued throughout the complex ν plane. When it is analytically continued to $\nu > 3$ it becomes of subleading order. This behavior is depicted in Fig. 4. Exactly the converse situation applies to the function $F^{>}(\nu;\epsilon)$.

Therefore in the neighborhood of $\nu = 3$,

$$F(\nu;\epsilon) = F^{>}(\nu;\epsilon) + F^{<}(\nu;\epsilon) = A^{>}(\nu)\epsilon^{3-\nu} + A^{<}(\nu)\epsilon^{\nu-3} \tag{4.28}$$

is accurate to leading and subleading order in ϵ. For $\nu > 3$, the term with the coefficient $A^{>}(\nu)$ is dominant while that with the coefficient $A^{<}(\nu)$ is subdominant. For $\nu < 3$ the roles of the the dominant and subdominant terms are interchanged. It should be emphasized that the addition of the two terms contains no double counting since in each region one term, and one term only, dominates. Since the theory is well defined at the physical dimension $\nu = 3$, the poles must cancel, which requires that

$$R^{>} = R^{<}. \tag{4.29}$$

Using

$$\epsilon^{\pm(\nu-3)} = e^{\pm(\nu-3)\ln\epsilon}, \tag{4.30}$$

we now have, in the neighborhood of $\nu = 3$,

$$\begin{aligned} F(\nu;\epsilon) &= \frac{R^{>}}{\nu - 3}\left[e^{+(\nu-3)\ln\epsilon} - e^{-(\nu-3)\ln\epsilon}\right] + r^{>} + r^{<} \\ &= 2R^{>}\ln\epsilon + (r^{>} + r^{<}). \end{aligned} \tag{4.31}$$

We must emphasize that this method of dimensional continuation provides not only the coefficient $2R$ out in front of $\ln\epsilon$ (which is often not too difficult to compute), but the constant $r^{>} + r^{<}$ in addition to this logarithm (which is often difficult to compute).

A relevant example is provided by the simple model of the energy loss presented in the previous subsection. According to our general method, in the neighborhood of $\nu=3$ we must have

$$I(\nu) = I^{>}(\nu) + I^{<}(\nu) = \frac{b_{\min}^{3-\nu}}{\nu-3} + \frac{b_{\max}^{3-\nu}}{3-\nu}. \tag{4.32}$$

The $\nu \to 3$ limit produces

$$I(\nu) = \frac{b_{\max}^{3-\nu}}{\nu-3}\left[\left(\frac{b_{\min}}{b_{\max}}\right)^{3-\nu} - 1\right] \to -\ln\left\{\frac{b_{\min}}{b_{\max}}\right\}. \tag{4.33}$$

This is precisely the value (4.1) of the familiar impact integral evaluated directly in three dimensions.

Another instructive example of this method is provided by an examination of the modified Bessel function $K_\nu(z)$ for small ν and small z. This is discussed in Refs. [3,1], and in more detail in Ref. [2]. These works should be consulted if the explanation already given is not convincing.

Although we have sketched the basic idea of our dimensional continuation scheme leading to the result (4.31), in this paper we shall apply it in a slightly different form, a form similar to the more physical example that led to the result (4.33). As we have explained, the electrostatic potential in ν spatial dimensions has the functional form $e/r^{\nu-2}$ so that the energy between two point charges a distance r apart is proportional to $e^2/r^{\nu-2}$. As we shall see explicitly in our work below, the electron-ion energy exchange rate contains an overall dimension-bearing factor $\beta_e e^2$. This factor has the dimensions of length to the power $\nu-2$. To define a quantity whose physical dimension does not vary as the spatial dimension varies, the factor $\beta_e e^2$ must be accompanied by a factor of length raised to the power $3-\nu$, which gives a result that has a constant factor of 1/length in all spatial dimensions ν. For the $\nu>3$ contribution, a scattering term with a length cutoff given by the electron thermal wavelength λ_e, the needed dimensional factor is given by $\lambda_e^{3-\nu}$ as we shall explicitly find below. For the $\nu<3$ contribution, a long-distance Debye screened interaction term, the needed dimensional factor is given by $(1/\kappa_e)^{3-\nu}$ as we shall also explicitly see below. Thus in all spatial dimensions near $\nu=3$, the rate has the structure

$$G(\nu) = \beta_e e^2\left[\lambda_e^{3-\nu} B^{>}(\nu) + \left(\frac{1}{\kappa_e}\right)^{3-\nu} B^{<}(\nu)\right], \tag{4.34}$$

and for ν near $\nu=3$,

$$B^{>}(\nu) = \frac{R}{\nu-3} + b_{>} \tag{4.35}$$

and

$$B^{<}(\nu) = \frac{R}{3-\nu} + b_{<}. \tag{4.36}$$

Writing $B=b_{>}+b_{<}$, we find that for ν near $\nu=3$,

$$G(\nu) = \beta_e e^2 \lambda_e^{3-\nu}\left[\frac{R}{\nu-3}\{1-(\lambda_e\kappa_e)^{\nu-3}\} + B\right]$$
$$\to \beta_e e^2[-R\ln\{\lambda_e\kappa_e\} + B], \tag{4.37}$$

in which the final line gives the $\nu=3$ limit.

An objection could be raised that we have not shown explicitly that larger subleading terms are not present. We have extracted terms that have the generic behavior $\epsilon^{3-\nu}$ for $\nu>3$ and $\epsilon^{\nu-3}$ for $\nu<3$. One might ask if there are additional terms with a power law dependence between $\epsilon^{3-\nu}$ and $\epsilon^{\nu-3}$. However, simple dimensional analysis shows that such terms of intermediate order cannot appear. The point is that the physics involves *only two* different mechanisms that dominate at large and small scales. These two different mechanisms involve different combinations of basic physical parameters and hence give quite different dependencies on the small parameter when the spatial dimension ν departs form $\nu=3$.

V. CONVERGENT KINETIC EQUATIONS

A number of authors [10–13] have proposed various versions of plasma kinetic equations that have neither short nor long range divergences. This work is summarized in the book of Liboff [14], which we shall outline here and then relate to our method of dimensional regularization.

Liboff, in his Eq. (2.75), writes the transport equation for a homogeneous system such as we consider as

$$\frac{\partial f}{\partial t} = B_0 + L_0 - \bar{R}, \tag{5.1}$$

where B_0 is the Boltzmann collision integral, L_0 is the Lenard-Balescu integral, and $\bar{R}$ is a renormalization term that cancels the singularities in B_0 and L_0 [15]. We should note that starting off with admittedly infinite and therefore undefined quantities, as in Ref. [14], is at best a heuristic procedure. This is to be contrasted with the renormalization procedure performed in modern quantum field theories where the starting point is a rigorously defined, finite theory because the starting point is a *regularized* theory. At any rate, the infinite renormalization term $\bar{R}$ is expressed formally as a double integral over both impact parameters and Fourier wave numbers. The integral over impact parameters b is broken up into a large impact parameter part $b>b_0$ and a small impact parameter part $b<b_0$, $\bar{R}=\bar{R}(>b_0)+\bar{R}_0(<b_0)$. It is then shown that $\bar{R}$ has a formal construction such that both $B_0-\bar{R}(>b_0)$ and $L_0-\bar{R}(<b_0)$ are finite.

The transport equation is thus rendered finite. Liboff concludes, "So we find that the combination of collision integrals gives a reasonable model for a convergent plasma kinetic equation." Our goal, however, is not just to find a "reasonable model," but to *calculate* the Coulomb logarithm in a precise and rigorous fashion. To leading order in the plasma density, we shall not only compute the coefficient out in front of the logarithm, but also the constants that appear in addition to the logarithm. See Ref. [2] for more details.

Although Gould and DeWitt [12] also separate the right-hand side of the transport equation into three terms, they do

so in such a fashion that each of the terms is finite and well defined. As shown in Appendix B of BPS [1], the formulation of Gould and DeWitt correctly gives the constant term as well as the leading Coulomb logarithm, and as far as these terms are concerned, their work is mathematically equivalent to our method of dimensional continuation. Both are accurate to $\mathcal{O}(g^2)$ in the plasma coupling, and no better. The trouble with their formulation is that it also produces a subset of higher order terms, and there is no reason that these provide a more accurate evaluation. As is well known, the inclusion of partial subsets of higher-order terms can sometimes give less rather than more accurate results.

VI. ENERGY AND TEMPERATURE RATES

The rate of change in the electron energy density transported to all the ions species vanishes when the two subsystems have the same temperature. Hence we may write

$$\frac{d\mathcal{E}_{eI}}{dt} = -\mathcal{C}_{eI}(T_e - T_I). \tag{6.1}$$

Since energy flows from the electrons to the ions when the electrons are hotter than the ions, $\mathcal{C}_{eI}$ is positive. Since the total energy is conserved, the rate at which energy is transferred from the ions to the electrons, $d\mathcal{E}_{Ie}/dt$, has the same coefficient $\mathcal{C}_{eI}$ but an overall sign change or, equivalently,

$$\frac{d\mathcal{E}_{Ie}}{dt} = -\mathcal{C}_{eI}(T_I - T_e). \tag{6.2}$$

A change in the energy of a subsystem in the plasma produces a corresponding change in the temperature of that subsystem. Thus for the electrons,

$$\Delta\mathcal{E}_{eI} = c_e \Delta T_e, \tag{6.3}$$

while for the ions

$$\Delta\mathcal{E}_{Ie} = \sum_i \Delta\mathcal{E}_i = c_I \Delta T_I. \tag{6.4}$$

Here, since the plasma interactions do not change particle number densities, the specific heats c_e and c_I are those at constant volume. Since $\Delta\mathcal{E}_{eI}$ is an energy density, these are the specific heats per unit volume. For a hot plasma that is not strongly coupled, the case treated in this paper, these specific heats are given by the familiar ideal gas results:

$$c_e = 3n_e/2 \text{ and } c_I = 3n_I/2, \tag{6.5}$$

where n_I is the total ionic density, the number of all the ions per unit volume. Thus Eqs. (6.1) and (6.2) are equivalent to

$$\frac{dT_e}{dt} = -\gamma_{eI}(T_e - T_I), \tag{6.6}$$

with $\gamma_{eI} = \mathcal{C}_{eI}/c_e$, and

$$\frac{dT_I}{dt} = -\gamma_{Ie}(T_I - T_e), \tag{6.7}$$

with $\gamma_{Ie} = \mathcal{C}_{eI}/c_I$. Moreover, the rate at which the separate temperatures approach one another is given by

$$\frac{d(T_e - T_I)}{dt} = -\Gamma(T_e - T_I), \tag{6.8}$$

in which

$$\Gamma = \mathcal{C}_{eI}\left(\frac{1}{c_e} + \frac{1}{c_I}\right). \tag{6.9}$$

We turn now to compute the rate coefficient $\mathcal{C}_{eI}$.

VII. BOLTZMANN EQUATION: SHORT-DISTANCE PHYSICS

We first work in $\nu > 3$ dimensions where the short-distance physics dominates. Thus the rate of change of the electron distribution is described by the Boltzmann equation with Fermi-Dirac statistics for the electrons and Maxwell-Boltzmann statistics for the heavy ions. The Boltzmann equation for the electron distribution, including the Pauli blocking of the scattered electrons and two-body quantum scattering effects, reads

$$\begin{aligned}\frac{\partial f_e(\mathbf{p}_e)}{\partial t} = \sum_i \int \frac{d^\nu p_e'}{(2\pi\hbar)^\nu}\frac{d^\nu p_i'}{(2\pi\hbar)^\nu}\frac{d^\nu p_i}{(2\pi\hbar)^\nu}|T|^2(2\pi\hbar)^\nu \\ \times \delta^\nu(\mathbf{p}_e' + \mathbf{p}_i' - \mathbf{p}_e - \mathbf{p}_i)(2\pi\hbar)\delta\left(\frac{\mathbf{p}_e'^2}{2m_e} + \frac{\mathbf{p}_i'^2}{2m_i}\right. \\ \left. - \frac{\mathbf{p}_e^2}{2m_e} - \frac{\mathbf{p}_i^2}{2m_i}\right)\{f_e(\mathbf{p}_e')f_i(\mathbf{p}_i')[1 - f_e(\mathbf{p}_e)] \\ - f_e(\mathbf{p}_e)f_i(\mathbf{p}_i)[1 - f_e(\mathbf{p}_e')]\},\end{aligned} \tag{7.1}$$

where T is the amplitude for the two-body scattering collision $ei \to e'i'$, and we have omitted the spatial convection term on the left-hand side because we are concerned with spatially uniform plasmas. Since the electrons are in thermal equilibrium with each other, there is no electron-electron interaction contribution to this time derivative. The electron kinetic energy density—the electron energy per unit volume—has the same form as the number density (3.5) save that an additional factor of $E_e(\mathbf{p}) = p^2/2m_e$ appears in the integrand. Hence the rate at which this energy density changes because of the electron ion interactions is given by

$$\frac{\partial\mathcal{E}_{eI}}{\partial t} = 2\int \frac{d^\nu p_e}{(2\pi\hbar)^\nu}\frac{p_e^2}{2m_e}\frac{\partial f_e(\mathbf{p}_e)}{\partial t}, \tag{7.2}$$

where the factor of 2 multiplying the integral accounts for the electron spin degeneracy. Using the crossing symmetry $\mathbf{p}_e \leftrightarrow \mathbf{p}_e'$ and $\mathbf{p}_i \leftrightarrow \mathbf{p}_i'$ of the scattering amplitude T in Eq. (7.1), the rate of energy exchange from the electrons to the ions (7.2) can be written as

$$\begin{aligned}\frac{\partial\mathcal{E}_{eI}^{>}}{\partial t} = 2\sum_i \int \frac{d^\nu p_e'}{(2\pi\hbar)^\nu}\frac{d^\nu p_i'}{(2\pi\hbar)^\nu}\frac{d^\nu p_e}{(2\pi\hbar)^\nu}\frac{d^\nu p_i}{(2\pi\hbar)^\nu}|T|^2 \\ \times (2\pi\hbar)^\nu\delta^\nu(\mathbf{p}_e' + \mathbf{p}_i' - \mathbf{p}_e - \mathbf{p}_i)(2\pi\hbar)\delta\left(\frac{\mathbf{p}_e'^2 - \mathbf{p}_e^2}{2m_e}\right. \\ \left. + \frac{\mathbf{p}_i'^2 - \mathbf{p}_i^2}{2m_i}\right)\frac{\mathbf{p}_e'^2 - \mathbf{p}_e^2}{2m_e}f_e(\mathbf{p}_e)f_i(\mathbf{p}_i)[1 - f_e(\mathbf{p}_e')],\end{aligned} \tag{7.3}$$

where the factor of 2 in front of the sum is the spin degeneracy $\mathfrak{g}_e=2$ for electrons. We have placed a "greater than" superscript on the left-hand side of the equation since we are now computing the $\nu>3$ contribution. We start by performing the $\mathbf{p}_i'$ integration in Eq. (7.3), using the momentum conserving delta function to set

$$\mathbf{p}_i' = \mathbf{p}_e + \mathbf{p}_i - \mathbf{p}_e'. \tag{7.4}$$

Defining the momentum transfer by

$$\mathbf{q} \equiv \mathbf{p}_e' - \mathbf{p}_e = \mathbf{p}_i - \mathbf{p}_i', \tag{7.5}$$

and the average of the initial and final electron momenta by

$$\bar{\mathbf{p}} \equiv \frac{1}{2}[\mathbf{p}_e' + \mathbf{p}_e], \tag{7.6}$$

we can simplify Eq. (7.3) to read

$$\frac{\partial \mathcal{E}_{eI}^{>}}{\partial t} = 2\sum_i \int \frac{d^\nu p_e'}{(2\pi\hbar)^\nu}\frac{d^\nu p_e}{(2\pi\hbar)^\nu}\frac{d^\nu p_i}{(2\pi\hbar)^\nu}|T|^2(2\pi\hbar)\delta\left(\frac{1}{m_i}\mathbf{p}_i\cdot\mathbf{q} - \frac{1}{m_e}\bar{\mathbf{p}}\cdot\mathbf{q} - \frac{1}{2m_i}\mathbf{q}^2\right)\frac{1}{m_e}\bar{\mathbf{p}}\cdot\mathbf{q} f_e(\mathbf{p}_e)f_i(\mathbf{p}_i)[1-f_e(\mathbf{p}_e')]. \tag{7.7}$$

Since T is a two-body scattering amplitude, its general form can depend upon both the square of the momentum transfer $q^2=\mathbf{q}\cdot\mathbf{q}$ and the total center-of-mass energy $W=p^2/2m_{ei}$, where the relative momentum is given by $\mathbf{p}=m_e(\mathbf{v}_e-\mathbf{v}_i)$, with m_{ei} being the reduced electron-ion mass. It is the W dependence in $T=T(W,q^2)$ that renders the integrals in Eq. (7.7) difficult to calculate because W depends explicitly on $\mathbf{p}_i$. In Sec. 12 of Ref. [1], this calculation is performed to all orders in the Coulomb scattering. For the work here, we shall be less general and exploit the fact that the electron-ion mass ratio m_e/m_i is very small (so the reduced mass m_{ei} is almost equal to the electron mass m_e). We shall assume that the electron and ion temperatures are not orders of magnitude apart, a mild restriction in all practical applications, so that

$$\beta_e m_e \ll \beta_I m_i. \tag{7.8}$$

Under these circumstances, the thermal average electron velocity is much larger than the ion velocity, and to a very good approximation $|\mathbf{v}_e-\mathbf{v}_i|=|\mathbf{v}_e|$. Thus the quantum Coulomb parameter $\eta_e=ee_i/4\pi\hbar|\mathbf{v}_e-\mathbf{v}_i|$ that appears in the Boltzmann equation can be replaced by a Coulomb parameter that contains only the electron velocity, $\eta_e\to Z_i e^2/4\pi\hbar|\mathbf{v}_e|$, where we have written $e_i=Z_i e$. The size of this parameter is estimated by its thermal average, which we denote by an overline. We use the simple Maxwell-Boltzmann distribution to estimate this average. For this classical distribution, the thermal average of $1/\mathbf{v}_e^2$ is precisely m_e/T_e, and so

$$\overline{\eta_{ei}^2} \simeq Z_i^2\left(\frac{e^2}{4\pi\hbar}\right)^2\frac{m_e}{T_e} = Z_i^2 2\pi\eta_e^2, \tag{7.9}$$

where in the second equality we have used the previous definition (4.4) of the electron quantum Coulomb parameter η_e together with the definition (3.7) of the electron thermal wavelength λ_e. Another way to write this is

$$\overline{\eta_{ei}^2} \simeq Z_i^2\frac{2\epsilon_H}{T_e}, \tag{7.10}$$

where $\epsilon_H\simeq 13.6$ eV previously noted in Eq. (2.9) is the binding energy of the hydrogen atom. The result (7.10) demonstrates that η_{ei} is quite small for the elevated temperature range that concerns us. Hence the scattering amplitude in Eq. (7.7) can be calculated in the Born approximation [16],

$$T \simeq T_B(q^2) = \hbar\frac{ee_i}{q^2}, \tag{7.11}$$

a quantity that depends only upon the square of the momentum transfer q^2, and not on the center-of-mass energy W.

In the Born approximation, the initial ion momentum $\mathbf{p}_i$ appears only in the delta function and phase-space density explicitly shown in Eq. (7.7), and not in the amplitude $T_B(q^2)$, and so the integration over this momentum variable can be carried out. If it were not for the delta-function factor, the $\mathbf{p}_i$ integration would simply entail

$$\int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} f_i(\mathbf{p}_i) = \int \frac{d^\nu p_i}{(2\pi)^\nu}\exp\left\{-\beta_I\left[\frac{p_i^2}{2m_i}-\mu_i\right]\right\} = \lambda_i^{-\nu}e^{\beta_I\mu_i} = n_i/\mathfrak{g}_i. \tag{7.12}$$

Following the convention exhibited in Eq. (3.5), the species index i for ions implicitly includes spin degrees of freedom, and so the integration over a single $f_i(\mathbf{p}_i)$ produces $n_i/\mathfrak{g}_i$. The delta function in Eq. (7.7) removes one of the components of the p_i integration, which is equivalent to supplying an extra factor of λ_i and retaining a Maxwell-Boltzmann factor corresponding to the component of the momentum $\mathbf{p}_i$ along the direction of $\mathbf{q}$. Hence

$$\int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} f_i(\mathbf{p}_i)(2\pi\hbar)\delta\left(\frac{\mathbf{p}_i\cdot\mathbf{q}}{m_i} - \frac{\bar{\mathbf{p}}\cdot\mathbf{q}}{m_e} - \frac{q^2}{2m_i}\right) = \frac{1}{q}\frac{n_i}{\mathfrak{g}_i}\lambda_i m_i \exp\left\{-\frac{\beta_I}{2m_i q^2}\left(\frac{m_i}{m_e}\bar{\mathbf{p}}\cdot\mathbf{q}+\frac{q^2}{2}\right)^2\right\}. \tag{7.13}$$

We shall often denote the magnitude of the momentum transfer by $q=|\mathbf{q}|$, as we have done here. We now change the remaining two integration variables $\mathbf{p}_e'$ and $\mathbf{p}_e$ in Eq. (7.7) to the variables $\bar{\mathbf{p}}$ and $\mathbf{q}$ defined in Eqs. (7.5) and (7.6), a change that has a unit Jacobian. Since the electrons are described by the Fermi-Dirac distribution (3.2), the Pauli blocking term in Eq. (7.7) can be written as

$$1-f_e(\bar{\mathbf{p}}+\mathbf{q}/2) = e^{-\beta_e\mu_e}\exp\left\{\frac{\beta_e}{2m_e}\left(\bar{\mathbf{p}}+\frac{1}{2}\mathbf{q}\right)^2\right\}f_e(\bar{\mathbf{p}}+\mathbf{q}/2). \tag{7.14}$$

Using these results, and neglecting terms involving the very small ratios m_e/m_i and $\beta_e m_e/\beta_I m_i$, we find that

$$\frac{\partial \mathcal{E}_{eI}^{>}}{\partial t} = 2\sum_i \frac{n_i}{\mathfrak{g}_i} \int \frac{d^\nu \bar{p}}{(2\pi\hbar)^\nu} \frac{d^\nu q}{(2\pi\hbar)^\nu} |T_B(q^2)|^2 \frac{m_i \lambda_i}{m_e} e^{-\beta_e \mu_e} \times f_e(\bar{\mathbf{p}} - \mathbf{q}/2) f_e(\bar{\mathbf{p}} + \mathbf{q}/2) \exp\left\{ + \frac{\beta_e}{2m_e}\left[\bar{p}_\perp^2 + \frac{1}{4}q^2 \right]\right\} \times \bar{\mathbf{p}} \cdot \hat{\mathbf{q}} \exp\left\{ - \frac{\beta_I}{2m_e}\frac{m_i}{m_e}\left[\bar{\mathbf{p}} \cdot \hat{\mathbf{q}} + \frac{m_e}{2m_i}\left(1 - \frac{\beta_e}{\beta_I}\right) q \right]^2 \right\}, \tag{7.15}$$

where $\hat{\mathbf{q}} = \mathbf{q}/|\mathbf{q}|$, the variable $\bar{\mathbf{p}}_\perp$ in the first exponent is the component of $\bar{\mathbf{p}}$ orthogonal to the momentum transfer $\mathbf{q}$, so that $\bar{\mathbf{p}} = \bar{\mathbf{p}}_\perp + (\bar{\mathbf{p}} \cdot \hat{\mathbf{q}})\hat{\mathbf{q}}$ with $\bar{\mathbf{p}}_\perp \cdot \mathbf{q} = 0$.

We can simplify the rate (7.15) by further exploiting the consequences of the very small ratio m_e/m_i. In the order of magnitude estimates that follow, we will use the symbol β to designate the inverse temperatures of either the electrons or the ions. This is possible because the temperature disparity is not very severe. We now see that the thermal distribution functions in the second line of Eq. (7.15) restrict the size of the momenta to be of the order

$$\bar{p}_\perp^2 \sim \frac{m_e}{\beta} \text{ and } q^2 \sim \frac{m_e}{\beta}. \tag{7.16}$$

For the longitudinal component of the electron momentum, the form of the exponential in the last line of Eq. (7.15) motivates the change of variables to

$$\bar{p}'_\parallel \equiv \bar{\mathbf{p}} \cdot \hat{\mathbf{q}} + \frac{m_e}{2m_i}\left(1 - \frac{\beta_e}{\beta_I}\right) q. \tag{7.17}$$

Under this change of variables, the last line in Eq. (7.15) becomes

$$\left[\bar{p}'_\parallel - \frac{m_e}{2m_i}\left(1 - \frac{\beta_e}{\beta_I}\right) q \right] \exp\left\{ - \frac{\beta_I}{2m_e}\frac{m_i}{m_e} \bar{p}_\parallel'^2 \right\}, \tag{7.18}$$

a term whose exponent restricts the size of the longitudinal component to be

$$|\bar{p}'_\parallel| \sim \sqrt{\frac{m_e^2}{\beta m_i}} \sim \sqrt{\frac{m_e}{m_i}} \bar{p}_\perp \sim \sqrt{\frac{m_e}{m_i}} q. \tag{7.19}$$

This means that the second term in square brackets at the start of expression (7.18), the term $(m_e/m_i)q \sim \sqrt{m_e^3/\beta m_i^2}$, is a factor $\sqrt{m_e/m_i}$ smaller than the first term $\bar{p}'_\parallel$. However, as we shall find, the first term integrates identically to zero, leaving the ostensibly smaller second term as the leading order contribution. To see this, we first note that the electron distributions $f_e(\bar{\mathbf{p}} \mp \mathbf{q}/2)$ in Eq. (7.15) are functions of the dimensionless variables

$$\beta_e E_e(\bar{\mathbf{p}} \mp \mathbf{q}/2) = \frac{\beta_e}{2m_e}(\bar{\mathbf{p}} \mp \mathbf{q}/2)^2. \tag{7.20}$$

Here, we must express the old variable $\bar{\mathbf{p}}$ in terms of the new variable

$$\bar{\mathbf{p}}' \equiv \bar{\mathbf{p}}_\perp + \bar{p}'_\parallel \hat{\mathbf{q}}, \tag{7.21}$$

or in terms of the vectors (7.5) and (7.6),

$$\bar{\mathbf{p}}' = \bar{\mathbf{p}} + \frac{m_e}{2m_i}\left(1 - \frac{\beta_e}{\beta_i}\right)\mathbf{q}. \tag{7.22}$$

Then from Eq. (7.22), we see that replacing the old variable $\bar{\mathbf{p}}$ in Eq. (7.20) by the new variable $\bar{\mathbf{p}}'$ incurs relative error of order

$$(m_e/m_i)\bar{\mathbf{p}}' \cdot \mathbf{q}(1/\bar{p}^2) \sim (m_e/m_i)^{3/2}, \tag{7.23}$$

an error beyond the leading term that we retain. That is to say, we can simply replace

$$f_e(\bar{\mathbf{p}} - \mathbf{q}/2) f_e(\bar{\mathbf{p}} + \mathbf{q}/2) \to f_e(\bar{\mathbf{p}}' - \mathbf{q}/2) f_e(\bar{\mathbf{p}}' + \mathbf{q}/2). \tag{7.24}$$

This product is explicitly even in $\mathbf{q}$, as are the remaining terms in the integrand, and consequently, the odd term $\bar{p}'_\parallel$ in the prefactor of Eq. (7.18) integrates to zero. The energy rate (7.15) now reduces to

$$\frac{\partial \mathcal{E}_{eI}^{>}}{\partial t} = -2\sum_i \frac{n_i}{\mathfrak{g}_i} \int \frac{d^\nu \bar{p}'}{(2\pi\hbar)^\nu} \frac{d^\nu q}{(2\pi\hbar)^\nu} |T_B(q^2)|^2 \frac{m_i \lambda_i}{m_e} e^{-\beta_e \mu_e} \times f_e(\bar{\mathbf{p}}' - \mathbf{q}/2) f_e(\bar{\mathbf{p}}' + \mathbf{q}/2) \exp\left\{ + \frac{\beta_e}{2m_e}\left[\bar{p}_\perp^2 + \frac{1}{4}q^2 \right]\right\} \times \frac{m_e}{2m_i}\left(1 - \frac{\beta_e}{\beta_I}\right) q \exp\left\{ - \frac{\beta_I}{2m_e}\frac{m_i}{m_e} \bar{p}_\parallel'^2 \right\}. \tag{7.25}$$

The integral over the momentum $\bar{\mathbf{p}}' = \bar{\mathbf{p}}_\perp + \bar{p}'_\parallel \hat{\mathbf{q}}$ contains $\nu - 1$ integrals from $\bar{\mathbf{p}}_\perp$ and one integral from $\bar{p}'_\parallel$.

Now that the leading contribution has be extracted, we can make further reductions by omitting several more terms in m_e/m_i. In particular, we may now neglect the longitudinal part $\bar{p}'_\parallel = \bar{\mathbf{p}}' \cdot \hat{\mathbf{q}}$ relative to $q = |\mathbf{q}|$ in the electron distribution functions $f_e(\bar{\mathbf{p}}' \mp \mathbf{q})$, which then become functions only of $(\bar{\mathbf{p}}_\perp \mp \mathbf{q}/2)^2$. In fact, since $\bar{\mathbf{p}}_\perp \cdot \mathbf{q} = 0$, both electron distribution functions have the same argument,

$$\beta_e E_e(\bar{\mathbf{p}}_\perp \pm \mathbf{q}/2) = \frac{\beta_e}{2m_e}\left(\bar{p}_\perp^2 + \frac{1}{4}q^2 \right), \tag{7.26}$$

and their product becomes a simple square: $f_e(\bar{\mathbf{p}}' - \mathbf{q}/2) f_e(\bar{\mathbf{p}}' + \mathbf{q}/2) = [f_e(\bar{\mathbf{p}}_\perp + \mathbf{q}/2)]^2$. The longitudinal component $\bar{p}'_\parallel$ now appears only in the final factor of the integrand in Eq. (7.25), and we may therefore explicitly perform the integration over this part of the momentum,

$$\int_{-\infty}^{\infty} \frac{d\bar{p}'_\parallel}{2\pi\hbar} \exp\left\{ - \frac{\beta_I}{2m_i}\left(\frac{m_i}{m_e}\right)^2 \bar{p}_\parallel'^2 \right\} = \frac{m_e}{m_i \lambda_i}, \tag{7.27}$$

where the ionic thermal wavelength λ_i is determined from Eq. (3.7). We can now express the rate (7.25) as

$$\frac{\partial \mathcal{E}_{eI}^{>}}{\partial t} = -m_e\left(1-\frac{\beta_e}{\beta_I}\right)\sum_i \frac{n_i}{\mathfrak{g}_i m_i} \times \int \frac{d^{\nu-1}\bar{p}_\perp}{(2\pi\hbar)^{\nu-1}}\frac{d^\nu q}{(2\pi\hbar)^\nu}|T_B(q^2)|^2 q e^{-\beta_e\mu_e} \times [f_e(\bar{\mathbf{p}}_\perp+\mathbf{q}/2)]^2 \exp\left\{+\frac{\beta_e}{2m_e}\left[\bar{p}_\perp^2+\frac{1}{4}q^2\right]\right\}. \tag{7.28}$$

The integration over the momentum transfer $\mathbf{q}$ is damped at large values of $q=|\mathbf{q}|$, because at such large values

$$q\to\infty: e^{-\beta_e\mu_e}[f_e(\bar{\mathbf{p}}_\perp+\mathbf{q}/2)]^2 \exp\left\{+\frac{\beta_e}{2m_e}\left[\bar{p}_\perp^2+\frac{1}{4}q^2\right]\right\} \to e^{+\beta_e\mu_e}\exp\left\{-\frac{\beta_e}{2m_e}\left[\bar{p}_\perp^2+\frac{1}{4}q^2\right]\right\}. \tag{7.29}$$

Since the limit (7.29) constrains the integrand to small q, this further supports the use of the Born approximation (7.11), which allows us to express the rate (7.28) as

$$\frac{\partial \mathcal{E}_{eI}^{>}}{\partial t} = -\beta_e e^2 m_e \hbar^2 \omega_I^2 (T_e - T_I)\int \frac{d^{\nu-1}\bar{p}_\perp}{(2\pi\hbar)^{\nu-1}} \times \frac{d^\nu q}{(2\pi\hbar)^\nu}\frac{1}{q^3}e^{-\beta_e\mu_e}[f_e(\bar{\mathbf{p}}_\perp+\mathbf{q}/2)]^2 \times \exp\left\{+\frac{\beta_e}{2m_e}\left[\bar{p}_\perp^2+\frac{1}{4}q^2\right]\right\}, \tag{7.30}$$

where ω_I^2 is the sum over all the ionic species of the squared plasma frequencies,

$$\omega_I^2 = \sum_{\text{ion species}} \omega_i^2 = \sum_i \frac{e_i^2 n_i}{m_i}. \tag{7.31}$$

Now that the clutter has abated, we can more easily study the nature of the parameters that enter into the $\nu>3$ contribution of the energy exchange rate. Dividing the time derivative of the electron energy density by the electron specific heat $3n_e/2$ gives the rate of the electron temperature change already noted in Eq. (6.7), namely,

$$\frac{\partial T_e}{\partial t} = -\gamma_{eI}(T_e - T_I). \tag{7.32}$$

The integral of each momentum, with the normalizing denominator $2\pi\hbar$, gives a pure number times a factor of $1/\lambda_e$. Since $n_e\sim\lambda_e^{-\nu}$, and each factor of the momentum transfer $|\mathbf{q}|$ will produce a factor of $\hbar/\lambda_e$, we conclude from Eq. (7.30) that

$$\gamma_{eI}\sim\beta_e e^2 m_e\hbar^2\omega_I^2\lambda_e^{1-\nu}(\lambda_e/\hbar)^3 \sim \left(\frac{e^2}{\lambda_e^{\nu-2}}\frac{1}{T_e}\right)\left(\frac{\hbar\omega_I}{T_e}\right)\omega_I, \tag{7.33}$$

where in the second line we have made use of

$$\lambda_e^2\sim\hbar^2\beta_e/m_e \text{ and } \beta_e = 1/T_e. \tag{7.34}$$

In a ν-dimensional space, the energy between two electrons a distance λ_e apart is, up to a constant, given by $e^2/\lambda_e^{\nu-2}$. Hence the first factor in parentheses in the last line above is dimensionless. Since $\hbar\omega_I$ is an energy, the second factor is also dimensionless. Thus the overall dimension of γ_{eI} is that of the final factor ω_I, the correct dimension of an inverse time or rate. Although a factor of $\hbar$ appears here, it is canceled by the single factor of $\hbar$ that appears in $1/\lambda_e^{\nu-2}$ in the $\nu\to3$ limit, and so in this limit the rate is a classical quantity. However, as we shall see, the dimensional continuation method that we use produces logarithms, and a logarithm of $\hbar$ will appear in the final result. Finally, we should note that the rate involves the first power of the ion density, a power that does not depend upon the spatial dimensionality ν.

The rate (7.30) for $\nu>3$ diverges when $\nu\to3^+$, a divergence that is canceled by the $\nu\to3^-$ limit of the rate for $\nu<3$ that we compute in the next section. This latter rate involves purely classical dynamics. Thus it entails a wave number $\mathbf{k}$ that comes from the Fourier transform of a potential which is the analog of the quantum momentum transfer $\mathbf{q}$, but with $\mathbf{q}=\hbar\mathbf{k}$. With this replacement, the electron distributions would become $f_e(\bar{\mathbf{p}}'+\hbar\mathbf{k}/2)$, but since only classical quantities appear in the forthcoming $\nu<3$ contribution, in this part the electron distributions must appear only as $f_e(\bar{\mathbf{p}}')$. Thus to separate out a part of the $\nu>3$ Boltzmann expression for the rate that will combine in a simple fashion with the $\nu<3$ contribution that we shall soon examine, we construct this part by making the replacement

$$[f_e(\bar{\mathbf{p}}_\perp+\mathbf{q}/2)]^2\exp\left\{+\frac{\beta_e}{8m_e}q^2\right\} \to [f_e(\bar{\mathbf{p}}_\perp)]^2\exp\left\{-\frac{\beta_e}{8m_e}q^2\right\}, \tag{7.35}$$

which exhibits the needed large q^2 damping given in the limit (7.29). Accordingly, we decompose the rate of energy transfer into a potentially singular part and a regular part,

$$\frac{\partial \mathcal{E}_{eI}^{>}}{\partial t} = \frac{\partial \mathcal{E}_{eI}^{>S}}{\partial t} + \frac{\partial \mathcal{E}_{eI}^{>R}}{\partial t}, \tag{7.36}$$

where

$$\frac{\partial \mathcal{E}_{eI}^{>S}}{\partial t} = -\beta_e e^2 m_e\hbar^2\omega_I^2(T_e-T_I)\int \frac{d^{\nu-1}\bar{p}_\perp}{(2\pi\hbar)^{\nu-1}}\frac{d^\nu q}{(2\pi\hbar)^\nu}\frac{1}{q^3} \times \exp\left\{-\frac{\beta_e}{8m_e}q^2\right\}e^{-\beta_e\mu_e}[f_e(\bar{\mathbf{p}}_\perp)]^2\exp\left\{+\frac{\beta_e}{2m_e}\bar{p}_\perp^2\right\}, \tag{7.37}$$

and

$$\frac{\partial \mathcal{E}_{eI}^{>R}}{\partial t} = -\beta_e e^2 m_e\hbar^2\omega_I^2(T_e-T_I)\int \frac{d^2\bar{p}_\perp}{(2\pi\hbar)^2}\frac{d^3 q}{(2\pi\hbar)^3}\frac{1}{q^3}e^{-\beta_e\mu_e} \times \left(\left[f_e(\bar{\mathbf{p}}_\perp+\mathbf{q}/2)\right]^2\exp\left\{+\frac{\beta_e}{2m_e}\left(\bar{p}_\perp^2+\frac{1}{4}q^2\right)\right\} - [f_e(\bar{\mathbf{p}}_\perp)]^2\exp\left\{+\frac{\beta_e}{2m_e}\left(\bar{p}_\perp^2-\frac{1}{4}q^2\right)\right\}\right). \tag{7.38}$$

Here in the regular part, we have taken the $\nu \to 3$ limit since there is no impediment in doing so.

The singular part (7.37) may be simplified by performing the q integration. Passing to hyperspherical coordinates gives

$$\begin{aligned}\int \frac{d^\nu q}{(2\pi\hbar)^\nu}\frac{1}{q^3}\exp\left\{-\frac{\beta_e}{8m_e}q^2\right\} &= \frac{\Omega_{\nu-1}}{(2\pi\hbar)^\nu}\int_0^\infty \frac{dq}{q}q^{\nu-3}\exp\left\{-\frac{\beta_e}{8m_e}q^2\right\} \\ &= \frac{\Omega_{\nu-1}}{(2\pi\hbar)^\nu}\frac{1}{2}\int_0^\infty \frac{dx}{x}\left(\frac{8m_e}{\beta_e}x\right)^{(\nu-3)/2}e^{-x} \\ &= \frac{\Omega_{\nu-1}}{(2\pi\hbar)^3}\frac{1}{2}\left(\frac{\pi\lambda_e^2}{4}\right)^{(3-\nu)/2}\Gamma\left(\frac{\nu-3}{2}\right),\end{aligned} \tag{7.39}$$

where $\Omega_{\nu-1}$ is the area of a unit $(\nu-1)$ sphere embedded in a ν-dimensional space. In the second line above we have made an obvious change to a dimensionless integration variable x, and in the last line we have identified the resulting integral with a standard representation of the gamma function. We thus have

$$\begin{aligned}\frac{\partial \mathcal{E}_{eI}^{>S}}{\partial t} = &-\frac{\beta_e e^2 m_e}{2\hbar}\omega_I^2\frac{\Omega_{\nu-1}}{(2\pi)^3}\left(\frac{\pi\lambda_e^2}{4}\right)^{(3-\nu)/2}\Gamma\left(\frac{\nu-3}{2}\right)(T_e-T_I) \\ &\times\int\frac{d^{\nu-1}\bar{p}_\perp}{(2\pi\hbar)^{\nu-1}}[f_e(\bar{\mathbf{p}}_\perp)]^2\exp\left\{\beta_e\left[\frac{\bar{p}_\perp^2}{2m_e}-\mu_e\right]\right\}.\end{aligned} \tag{7.40}$$

As we shall see, the $\nu<3$ contribution calculated in the next section contains the same integral over the transverse components $\bar{\mathbf{p}}_\perp$, so it will be convenient to perform this integral when we add these terms together.

The regular part of the energy exchange rate may also be simplified since the integral over the electron distribution functions can be performed when $\nu=3$. Namely, we pass to polar coordinates, with the angular integration simply producing a factor of 2π, to obtain

$$\begin{aligned}&\int\frac{d^2\bar{p}_\perp}{(2\pi\hbar)^2}[f_e(\bar{\mathbf{p}}_\perp)]^2\exp\left\{\beta_e\left[\frac{\bar{p}_\perp^2}{2m_e}-\mu_e\right]\right\} \\ &= \frac{m_e}{2\pi\beta_e\hbar^2}\int_0^\infty d\left(\frac{\beta_e p^2}{2m_e}\right)\frac{\exp\left\{+\beta_e\left[\frac{p^2}{2m_e}-\mu_e\right]\right\}}{\left[\exp\left\{+\beta_e\left[\frac{p^2}{2m_e}-\mu_e\right]\right\}+1\right]^2} \\ &= \frac{1}{\lambda_e^2}\frac{1}{\exp\{-\beta_e\mu_e\}+1}.\end{aligned} \tag{7.41}$$

Hence making the replacement

$$-\beta_e\mu_e \to \beta_e\left[\frac{q^2}{8m_e}-\mu_e\right] \tag{7.42}$$

for the first term, we have

$$\begin{aligned}\frac{\partial \mathcal{E}_{eI}^{>R}}{\partial t} = &-\frac{\beta_e e^2 m_e}{2\hbar}\frac{\omega_I^2}{\pi^2}\frac{1}{\lambda_e^2}(T_e-T_I) \\ &\times\int_0^\infty\frac{dq}{q}\left\{\frac{1}{\exp\{\beta_e[(q^2/8m_e)-\mu_e]\}+1}\right. \\ &\left.-\frac{\exp\{-\beta_e(q^2/8m_e)\}}{\exp\{-\beta_e\mu_e\}+1}\right\} \\ = &-\frac{\beta_e e^2 m_e}{2\hbar}\frac{\omega_I^2}{\pi^2}\frac{1}{\lambda_e^2}(T_e-T_I)\frac{1}{2}\int_0^\infty dx \\ &\times\ln x\left\{\frac{\exp\{x-\beta_e\mu_e\}}{[\exp\{x-\beta_e\mu_e\}+1]^2}\right. \\ &\left.-\frac{\exp\{-x\}}{\exp\{-\beta_e\mu_e\}+1}\right\},\end{aligned} \tag{7.43}$$

where the last line follows from a trivial change of integration variables and a partial integration.

VIII. LENARD-BALESCU EQUATION: LONG-DISTANCE PHYSICS

We turn now to calculate the leading order long-distance physics by working in spatial dimensions $\nu<3$. This is done by employing the Lenard-Balescu equation with Fermi-Dirac statistics for the electrons and Maxwell-Boltzmann statistics for the heavy ions. For the spatially homogeneous system that we work with, the Lenard-Balescu equation with the appropriate Pauli blocking reads

$$\begin{aligned}\frac{\partial f_e(\mathbf{p}_e)}{\partial t} = &-\frac{\partial}{\partial\mathbf{p}_e}\cdot\sum_i\int\frac{d^\nu p_i}{(2\pi\hbar)^\nu}\frac{d^\nu k}{(2\pi)^\nu}\mathbf{k}\left|\frac{ee_i}{k^2\epsilon(k,\mathbf{k}\cdot\mathbf{v}_i)}\right|^2 \\ &\times\pi\delta(\mathbf{k}\cdot\mathbf{v}_e-\mathbf{k}\cdot\mathbf{v}_i)\left\{\mathbf{k}\cdot\frac{\partial f_i(\mathbf{p}_i)}{\partial\mathbf{p}_i}f_e(\mathbf{p}_e)\right. \\ &\left.\times[1-f_e(\mathbf{p}_e)]-f_i(\mathbf{p}_i)\mathbf{k}\cdot\frac{\partial f_e(\mathbf{p}_e)}{\partial\mathbf{p}_e}\right\},\end{aligned} \tag{8.1}$$

where the gradient $\partial/\partial\mathbf{p}_e$ acts on everything to its right, and $\epsilon(k,\omega)$ is the classical dielectric function for the plasma discussed in Appendix A. As was shown in Appendix C of Ref. [1], the usual nondegenerate Lenard-Balescu equation is a formal limit of the Boltzmann equation. The same methods that were employed there may be used to show that Eq. (8.1) is the corresponding long-distance equation when the electrons are degenerate and described by Fermi-Dirac statistics. If the electrons and ions are in equilibrium with themselves at temperatures T_e and T_i, respectively, then their distribution functions f_e and f_i are given by Eqs. (3.2) and (3.1), in which case the terms in curly braces can be written as

$$(\beta_e\mathbf{k}\cdot\mathbf{v}_e-\beta_i\mathbf{k}\cdot\mathbf{v}_i)f_i(\mathbf{p}_i)[f_e(\mathbf{p}_e)]^2\exp\left\{\beta_e\left[\frac{p_e^2}{2m_e}-\mu_e\right]\right\}. \tag{8.2}$$

Because the delta function equates $\mathbf{k}\cdot\mathbf{v}_e$ with $\mathbf{k}\cdot\mathbf{v}_i$, the factor (8.2) and with it the right-hand side of Eq. (8.1) vanish when

the electrons and ions are in thermal equilibrium with a common temperature T, the electrons being described by a Fermi-Dirac distribution and the ions by a Maxwell-Boltzmann distribution. This confirms the validity of Eq. (8.1), the Lenard Balescu equation with Pauli blocking for degenerate electrons.

Using Eq. (8.2) with the ions at the same inverse temperature $\beta_I = 1/T_I$, and upon integrating the total derivative $\partial/\partial \mathbf{p}_e$ by parts, we can express the energy exchange rate as

$$\frac{\partial \mathcal{E}_{eI}^{<}}{\partial t} = 2\sum_i \int \frac{d^\nu p_e}{(2\pi\hbar)^\nu}\frac{d^\nu p_i}{(2\pi\hbar)^\nu}\frac{d^\nu k}{(2\pi)^\nu}\mathbf{k}\cdot\mathbf{v}_e \left|\frac{ee_i}{k^2\epsilon(k,\mathbf{k}\cdot\mathbf{v}_i)}\right|^2 \times \pi\delta(\mathbf{k}\cdot\mathbf{v}_e - \mathbf{k}\cdot\mathbf{v}_i)(\beta_e\mathbf{k}\cdot\mathbf{v}_e - \beta_I\mathbf{k}\cdot\mathbf{v}_i) \times f_i(\mathbf{p}_i)[f_e(\mathbf{p}_e)]^2 \exp\left\{\beta_e\left[\frac{p_e^2}{2m_e} - \mu_e\right]\right\}. \tag{8.3}$$

We have placed a "less than" superscript on the left-hand side of Eq. (8.3) to remind ourselves that the calculation is performed in $\nu < 3$ using the Lenard-Balescu equation. The distribution functions constrain the velocities of the ions and the electrons to be of the order $v_i \sim \sqrt{T_I/m_i}$ and $v_e \sim \sqrt{T_e/m_e}$, respectively. Since an ion mass is so much greater than that of an electron, and the temperature disparity is never excessively large for cases of interest, the ions move much slower than the electrons,

$$v_i \ll v_e. \tag{8.4}$$

This restriction also follows from the previous condition (7.8), the condition that $\beta_e m_e \ll \beta_I m_i$. To compute the rate (8.3), we first decompose the electron momentum into perpendicular and longitudinal components relative to the direction specified by $\hat{\mathbf{k}}$, so that $\mathbf{p}_e = \mathbf{p}_\perp + p_\parallel \hat{\mathbf{k}}$ with $\hat{\mathbf{k}}\cdot\mathbf{p}_\perp = 0$ and $p_\parallel = \hat{\mathbf{k}}\cdot\mathbf{p}_e = m_e\hat{\mathbf{k}}\cdot\mathbf{v}_e$. The delta function in Eq. (8.3) can be used to remove the parallel component $p_\parallel$ of the electron momenta integration. Since $dp_\parallel/(2\pi\hbar) = (m_e/2\pi\hbar k)d(\mathbf{k}\cdot\mathbf{v}_e)$, the use of this delta function produces a factor $(m_e/2\pi\hbar k)$ and makes the replacement $p_\parallel \to m_e\hat{\mathbf{k}}\cdot\mathbf{v}_i$. In view of the limit (8.4), this replacement makes $p_\parallel$ much smaller than the magnitude of the perpendicular components of the electron momenta $p_\perp$. Hence we can simply replace $\mathbf{p} \to \mathbf{p}_\perp$ in the remainder of the integrand. We shall also find it convenient (the heart hath its reasons), to insert a factor of unity in the form

$$1 = \int_{-\infty}^{+\infty} dv\, \delta(v - \hat{\mathbf{k}}\cdot\mathbf{v}_i), \tag{8.5}$$

which allows us to express Eq. (8.3) as

$$\frac{\partial \mathcal{E}_{eI}^{<}}{\partial t} = -2\beta_e\frac{e^2 m_e}{2\pi\hbar}(T_e - T_I) \times \int \frac{d^{\nu-1}p_\perp}{(2\pi\hbar)^{\nu-1}}[f_e(\mathbf{p}_\perp)]^2 \exp\left\{\beta_e\left[\frac{p_\perp^2}{2m_e} - \mu_e\right]\right\} \times \int \frac{d^\nu k}{(2\pi)^\nu} k \int_{-\infty}^{+\infty} dv \frac{\pi v^2}{|k^2\epsilon(k,vk)|^2}\sum_i \beta_I e_i^2 \times \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} f_i(\mathbf{p}_i)\,\delta(\hat{\mathbf{k}}\cdot\mathbf{v}_i - v). \tag{8.6}$$

This integral can be further simplified by taking advantage of the analytic properties of the dielectric function $\epsilon(k,\omega)$, discussed in some detail in Appendix A. Repeating Eq. (A16) here for convenience, we see that a considerable portion of the integral simplifies because

$$\frac{\pi v}{|k^2\epsilon(k,vk)|^2}\sum_i \beta_I e_i^2 \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} f_i(\mathbf{p}_i)\,\delta(\hat{\mathbf{k}}\cdot\mathbf{v}_i - v) = -\frac{1}{2i}\left\{\frac{1}{k^2 + \kappa_e^2 + F_I(v)} - \frac{1}{k^2 + \kappa_e^2 + F_I(-v)}\right\}. \tag{8.7}$$

This result is the (unemotional) reason that the factor of unity in the form displayed in Eq. (8.5) was inserted in the integrand. Here κ_e is the electronic contribution to the Debye wave number, including the effects of Fermi-Dirac statistics, as expressed by Eq. (A6), while F_I is a complex-valued function defined by Eq. (A9). It is important to realize that the simplification (8.7) only occurs when the ion species are summed over. The function $F_I(z)$ is analytic over the upper half of the complex z plane, and has the asymptotic behavior

$$|z| \to \infty: \quad F_I(z) \to -\frac{\omega_I^2}{z^2}, \tag{8.8}$$

where ω_I is the total ionic plasma frequency defined above in Eq. (7.31). Since an explicit odd factor of v appears in the integrand, we can write the resulting integral over v in Eqs. (8.6) in the form

$$-\int_{-\infty}^{+\infty} dv \frac{v}{2i}\left\{\frac{1}{k^2 + \kappa_e^2 + F_I(v)} - \frac{1}{k^2 + \kappa_e^2 + F_I(-v)}\right\} = \lim_{V\to\infty} i\int_{-V}^{+V} dv \frac{v}{k^2 + \kappa_e^2 + F_I(v)}. \tag{8.9}$$

The delta function in Eq. (8.7) removes the longitudinal components of the ionic momenta, leaving a Maxwell-Boltzmann factor involving the velocity v. Hence the left-hand side of the integrand in Eq. (8.9) is damped in a Gaussian fashion for large $|v|$. This rapid damping results from a cancellation between the terms with $F_I(v)$ and $F_I(-v)$ that only happens when the two velocities are exactly the negative of one another. Hence when we simplify the integrand by taking advantage of the odd prefactor as was done on the right-hand side of Eq. (8.9), we must integrate between the exact same negative and positive limits, between $-V$ and $+V$, and only afterward take the limit $V \to \infty$. Since $F_I(z)$ is analytic in the upper-half z plane, the integral (8.9) may be evaluated by contour integral techniques. Let C_V be a semicircle of radius V centered at the origin of the complex z plane, with an orientation that starts at $+V$ and ends at $-V$. We can traverse a closed circuit by moving from $-V$ to $+V$ along the real axis, with the circuit completed back to $-V$ by traversing C_V. The contour integral around this closed circuit vanishes since it contains no interior singularities,

$$0 = \oint dz \frac{z}{k^2 + \kappa_e^2 + F_I(z)} = \int_{-V}^{V} dv \frac{v}{k^2 + \kappa_e^2 + F_I(v)} + \int_{C_V} dz \frac{z}{k^2 + \kappa_e^2 + F_I(z)}. \tag{8.10}$$

Hence the integral (8.9) is equal to the negative of the integral over the semicircle C_V starting at $+V$ and ending at $-V$. We can now take the limit $V \to \infty$ and use the asymptotic form (8.8) for F_I along C_V. Since

$$\int_{C_V} dz z = iV^2 \int_0^{\pi} d\theta e^{2i\theta} = 0, \tag{8.11}$$

the leading term of the expansion of the denominator in Eq. (8.9) yields a vanishing contribution, and the only nonvanishing term is given by first-order term with $F_I(z)$ replaced with its asymptotic form (8.8):

$$\lim_{V\to\infty} i \int_{-V}^{+V} dv \frac{v}{k^2 + \kappa_e^2 + F_I(v)} = i \frac{1}{[k^2 + \kappa_e^2]^2} \times \lim_{V\to\infty} \int_{C_V} dz z \left[-\frac{\omega_I^2}{z^2} \right] = \frac{\pi}{[k^2 + \kappa_e^2]^2} \omega_I^2. \tag{8.12}$$

Upon passing to hyperspherical coordinates to perform the k integration, we now arrive at

$$\frac{\partial \mathcal{E}_{eI}^{<}}{\partial t} = -\frac{\beta_e e^2 m_e}{\hbar} (T_e - T_I) \int \frac{d^{\nu-1} p_\perp}{(2\pi\hbar)^{\nu-1}} [f_e(\mathbf{p}_\perp)]^2 \times \exp\left\{ \beta_e \left[\frac{p_\perp^2}{2m_e} - \mu_e \right] \right\} \omega_I^2 \frac{\Omega_{\nu-1}}{(2\pi)^\nu} \times \int_0^{\infty} k^{\nu-1} dk k \frac{1}{[k^2 + \kappa_e^2]^2}. \tag{8.13}$$

Changing variables by $k = t^{1/2}\kappa_e$ places the k integration in the form of a standard representation of the Euler beta function [17], and so we have

$$\int_0^{\infty} dk \frac{k^\nu}{[k^2 + \kappa_e^2]^2} = \frac{1}{2} \kappa_e^{\nu-3} \frac{\Gamma[(\nu+1)/2]\Gamma[(3-\nu)/2]}{\Gamma(2)}. \tag{8.14}$$

Finally, we are now able to express the $\nu < 3$ form of the electron-ion energy exchange as

$$\frac{\partial \mathcal{E}_{eI}^{<}}{\partial t} = -\frac{\beta_e e^2 m_e}{2\hbar} \omega_I^2 \frac{\Omega_{\nu-1}}{(2\pi)^3} \left(\frac{\kappa_e}{2\pi} \right)^{\nu-3} \Gamma\left(\frac{\nu+1}{2} \right) \Gamma\left(\frac{3-\nu}{2} \right) \times (T_e - T_I) \int \frac{d^{\nu-1} p_\perp}{(2\pi\hbar)^{\nu-1}} [f_e(\mathbf{p}_\perp)]^2 \times \exp\left\{ \beta_e \left[\frac{p_\perp^2}{2m_e} - \mu_e \right] \right\}. \tag{8.15}$$

IX. ADDING THE RATES

The sum of the singular part (7.40) for the $\nu > 3$ contribution to the electron-ion energy exchange rate and the $\nu < 3$ part (8.15) that we have just computed is

$$\frac{\partial \mathcal{E}_{eI}^{>S}}{\partial t} + \frac{\partial \mathcal{E}_{eI}^{<}}{\partial t} = -\frac{\beta_e e^2 m_e}{2\hbar} \omega_I^2 \frac{\Omega_{\nu-1}}{(2\pi)^3} (T_e - T_I) \times \int \frac{d^{\nu-1} p_\perp}{(2\pi\hbar)^{\nu-1}} [f_e(\mathbf{p}_\perp)]^2 \exp\left\{ \beta_e \left[\frac{p_\perp^2}{2m_e} - \mu_e \right] \right\} \times \left(\frac{\pi\lambda_e^2}{4} \right)^{(3-\nu)/2} \left\{ \Gamma\left(\frac{\nu-3}{2} \right) + \left(\frac{\kappa_e^2 \lambda_e^2}{16\pi} \right)^{(\nu-3)/2} \Gamma\left(\frac{\nu+1}{2} \right) \Gamma\left(\frac{3-\nu}{2} \right) \right\}. \tag{9.1}$$

As must be the case, the expression in the final curly braces above is finite in the $\nu \to 3$ limit. To extract this limit, we use

$$\nu \to 3: \quad \Gamma\left(\frac{\nu-3}{2} \right) \to \frac{2}{\nu-3} - \gamma,$$

$$\Gamma\left(\frac{3-\nu}{2} \right) \to \frac{2}{3-\nu} - \gamma,$$

$$\Gamma\left(\frac{\nu+1}{2} \right) \to 1 - (1-\gamma)\frac{3-\nu}{2} \tag{9.2}$$

to evaluate the $\nu \to 3$ limit of the last line in Eq. (9.1):

$$\left[\frac{2}{\nu-3} - \gamma \right] + \left(\frac{\kappa_e^2 \lambda_e^2}{16\pi} \right)^{(\nu-3)/2} \left[\frac{2}{3-\nu} - 1 \right] \to \ln\left\{ \frac{16\pi}{\kappa_e^2 \lambda_e^2} \right\} - \gamma - 1. \tag{9.3}$$

Since this last factor is finite in the $\nu \to 3$ limit, we may now take the $\nu \to 3$ limit of all the other quantities in Eq. (9.1). The integral over the perpendicular momenta in this limit was evaluated previously in Eq. (7.41), and so we now have

$$\frac{\partial \mathcal{E}_{eI}^{>S}}{\partial t} + \frac{\partial \mathcal{E}_{eI}^{<}}{\partial t} = -\frac{\beta_e e^2 m_e}{2\hbar} \frac{\omega_I^2}{2\pi^2} \frac{1}{\lambda_e^2} \frac{1}{\exp\{-\beta_e \mu_e\} + 1} \times \left[\ln\left\{ \frac{16\pi}{\kappa_e^2 \lambda_e^2} \right\} - \gamma - 1 \right] (T_e - T_I). \tag{9.4}$$

To this we must add the remaining finite part (7.43) of the $\nu > 3$ contribution, namely,

$$\frac{\partial \mathcal{E}_{eI}^{>R}}{\partial t} = -\frac{\beta_e e^2 m_e}{2\hbar} \frac{\omega_I^2}{2\pi^2} \frac{1}{\lambda_e^2} \times (T_e - T_I) \int_0^{\infty} dx \ln x \left\{ \frac{\exp\{x - \beta_e \mu_e\}}{[\exp\{x - \beta_e \mu_e\} + 1]^2} - \frac{\exp\{-x\}}{\exp\{-\beta_e \mu_e\} + 1} \right\}. \tag{9.5}$$

Recalling that we have defined [Eq. (6.1)]

$$\frac{d\mathcal{E}_{eI}}{dt} = -\mathcal{C}_{eI}(T_e - T_I), \tag{9.6}$$

we have now calculated the rate coefficient to leading order in the plasma coupling and to all orders in the electron fugacity $z_e = e^{\beta_e\mu_e}$,

$$\mathcal{C}_{eI} = \frac{\beta_e e^2 m_e}{2\hbar}\frac{\omega_I^2}{\pi^2}\frac{1}{\lambda_e^2}\left\{\frac{1}{\exp\{-\beta_e\mu_e\}+1}\frac{1}{2}\left[\ln\left\{\frac{16\pi}{\kappa_e^2\lambda_e^2}\right\} - \gamma - 1\right]\right.$$
$$+\frac{1}{2}\int_0^\infty dx \ln x\left[\frac{\exp\{x-\beta_e\mu_e\}}{[\exp\{x-\beta_e\mu_e\}+1]^2}\right.$$
$$\left.\left.-\frac{\exp\{-x\}}{\exp\{-\beta_e\mu_e\}+1}\right]\right\}. \tag{9.7}$$

By expanding the denominators, it is easy to check that

$$\int_0^\infty dx \ln x\left[\frac{\exp\{x-\beta_e\mu_e\}}{[\exp\{x-\beta_e\mu_e\}+1]^2} - \frac{\exp\{-x\}}{\exp\{-\beta_e\mu_e\}+1}\right]$$
$$= \sum_{l=1}^{\infty}(-1)^{l+1}\ln\{l+1\}e^{(l+1)\beta_e\mu_e}, \tag{9.8}$$

which is an expansion in powers of the electron fugacity $z_e = e^{\beta_e\mu_e}$.

To place this result in a form that is easily compared to that of BPS [1], we use the definition (3.7) of the thermal wavelength and a slight manipulation to write

$$\mathcal{C}_{eI} = \frac{\omega_I^2}{2\pi}\sqrt{\frac{\beta_e m_e}{2\pi}}\left(\frac{2\beta_e e^2}{\lambda_e^3}e^{\beta_e\mu_e}\right)$$
$$\times\left\{\frac{1}{\exp\{\beta_e\mu_e\}+1}\frac{1}{2}\left[\ln\left\{\frac{16\pi}{\kappa_e^2\lambda_e^2}\right\} - \gamma - 1\right]\right.$$
$$\left.+\frac{1}{2}\sum_{l=1}^{\infty}(-1)^{l+1}\ln\{l+1\}e^{l\beta_e\mu_e}\right\}. \tag{9.9}$$

In the dilute limit in which Maxwell-Boltzmann statistics apply, the fugacity $\exp\{\beta_e\mu_e\}$ is very small. The number density approximation (3.12) gives

$$\frac{2}{\lambda_e^3}e^{\beta_e\mu_e} \simeq n_e\left[1 + \frac{1}{2^{3/2}}e^{\beta_e\mu_e}\right], \tag{9.10}$$

and we see that keeping the first correction in the fugacity yields

$$\mathcal{C}_{eI} \simeq \frac{\omega_I^2}{2\pi}\sqrt{\frac{\beta_e m_e}{2\pi}}(\beta_e e^2 n_e)\left(\left[1 - \left(1 - \frac{1}{2^{3/2}}\right)e^{\beta_e\mu_e}\right]\right.$$
$$\left.\times\frac{1}{2}\left[\ln\left\{\frac{16\pi}{\kappa_e^2\lambda_e^2}\right\} - \gamma - 1\right] + \frac{1}{2}e^{\beta_e\mu_e}\ln 2\right). \tag{9.11}$$

Again remembering the fugacity approximation (3.17), which we repeat here,

$$\kappa_e^2 \simeq \beta_e e^2 n_e\left[1 - \frac{1}{2^{3/2}}e^{\beta_e\mu_e}\right], \tag{9.12}$$

and the definitions

$$\lambda_e^2 = \frac{2\pi\hbar^2\beta_e}{m_e} \quad \text{and} \quad \omega_e^2 = \frac{e^2 n_e}{m_e}, \tag{9.13}$$

we find that

$$\frac{16\pi}{\kappa_e^2\lambda_e^2} \simeq \frac{8T_e^2}{\hbar^2\omega_e^2}\left[1 + \frac{1}{2^{3/2}}e^{\beta_e\mu_e}\right], \tag{9.14}$$

and thus

$$\mathcal{C}_{eI} \simeq \frac{\omega_I^2}{2\pi}\sqrt{\frac{\beta_e m_e}{2\pi}}(\beta_e e^2 n_e)\left(\left[1 - \left(1 - \frac{1}{2^{3/2}}\right)e^{\beta_e\mu_e}\right]\right.$$
$$\times\frac{1}{2}\left[\ln\left\{\frac{8T_e^2}{\hbar^2\omega_e^2}\right\} - \gamma - 1\right]$$
$$\left.+ e^{\beta_e\mu_e}\left[\frac{1}{2}\ln 2 + \frac{1}{2^{5/2}}\right]\right). \tag{9.15}$$

We may use $e^{\beta_e\mu_e} \simeq \lambda_e^3 n_e/2$ inside the curly braces of Eq. (9.15). It is easy to confirm that with the neglect of the fugacity corrections, this is in agreement with Eq. (12.12) of BPS [1] after that equation is corrected as mentioned in the Introduction.

ACKNOWLEDGMENT

We would like to thank George E. Cragg for reading the manuscript.

APPENDIX A: THE DIELECTRIC FUNCTION

In Sec. VIII, the calculation of the rate in $\nu < 3$ using the Lenard-Balescu equation made extensive use of the plasma dielectric function and its various properties. The classical dielectric function for a collisionless plasma is discussed in Ref. [18], for example, and the form of the result that we shall use reads

$$\epsilon(k,\omega) = 1 + \sum_b \frac{e_b^2}{k^2}\int \frac{d^\nu p_b}{(2\pi\hbar)^\nu}\frac{1}{\omega - \mathbf{k}\cdot\mathbf{v}_b + i\eta}\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_b}f_b(\mathbf{p}_b), \tag{A1}$$

with the prescription $\eta \to 0^+$ defining the correct retarded response. The degenerate electrons are described by the thermal Fermi-Dirac distribution (3.2), so

$$\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_e}f_e(\mathbf{p}_e) = -\beta_e\mathbf{k}\cdot\mathbf{v}_e\frac{e^{\beta_e(E_e-\mu_e)}}{[e^{\beta_e(E_e-\mu_e)}+1]^2}$$
$$= -\beta_e\mathbf{k}\cdot\mathbf{v}_e f_e(\mathbf{p}_e)[1 - f_e(\mathbf{p}_e)]. \tag{A2}$$

On the other hand, the ions are described by the Maxwell-Boltzmann distribution (3.1), which is simply the large chemical potential limit $-\beta\mu \gg 1$ of the Fermi-Dirac distribution. In this limit the Pauli blocking term is removed, $[1 - f(\mathbf{p})] \to 1$, and so

$$\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_i}f_i(\mathbf{p}_i) = -\beta_i\mathbf{k}\cdot\mathbf{v}_i f_i(\mathbf{p}_i). \tag{A3}$$

For the real plasma considered in the text, the ions equilibrate to a common temperature $T_I = 1/\beta_I$; however, for the

purposes of this appendix, we shall take each ion species i to have an individual inverse temperature β_i. For degenerate electrons and Maxwell-Boltzmann ions, the dielectric function (A1) may therefore be expressed as

$$\epsilon(k,\omega) = 1 - \sum_i \frac{\beta_i e_i^2}{k^2} \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} \frac{\mathbf{k}\cdot\mathbf{v}_i f_i(\mathbf{p}_i)}{\omega - \mathbf{k}\cdot\mathbf{v}_i + i\eta} - \frac{\beta_e e^2}{k^2}\cdot 2\int \frac{d^\nu p_e}{(2\pi\hbar)^\nu} \frac{\mathbf{k}\cdot\mathbf{v}_e f_e(\mathbf{p}_e)}{\omega - \mathbf{k}\cdot\mathbf{v}_e + i\eta}[1 - f_e(\mathbf{p}_e)]. \tag{A4}$$

The factor of two in the electron contribution arises from a sum over the two spin components of the electron.

The dielectric function in the Lenard-Balescu equation has the functional form $\epsilon(k,\mathbf{v}\cdot\mathbf{k})$, with the speed $|\mathbf{v}|$ much less than the electron thermal velocity. Hence in the electron contribution to the dielectric function, the magnitude of $\omega = \mathbf{k}\cdot\mathbf{v}$ is much less than the typical magnitude of $\mathbf{k}\cdot\mathbf{v}_e$, and we can use the $\omega \to 0$ limit in which

$$-\frac{\beta_e e^2}{k^2}\cdot 2\int \frac{d^\nu p_e}{(2\pi\hbar)^\nu} \frac{\mathbf{k}\cdot\mathbf{v}_e}{\omega - \mathbf{k}\cdot\mathbf{v}_e + i\eta} f_e(\mathbf{p}_e)[1 - f_e(\mathbf{p}_e)] \to \frac{\kappa_e^2}{k^2}, \tag{A5}$$

where

$$\kappa_e^2 = 2\beta_e e^2 \int \frac{d^\nu p_e}{(2\pi\hbar)^\nu} f_e(\mathbf{p}_e)[1 - f_e(\mathbf{p}_e)] \tag{A6}$$

defines the electron contribution to the squared Debye wave number, including the effects of Fermi-Dirac statistics which are explicitly exhibited by the Pauli blocking factor $[1 - f_e(\mathbf{p_e})]$. From the form (3.2) of the thermal Fermi-Dirac distribution $f_e(\mathbf{p}_e)$ and the definition (3.5) of the number density, we see that

$$\kappa_e^2 = e^2\beta_e \frac{\partial n_e}{\partial(\beta_e\mu_e)}. \tag{A7}$$

Remembering the structure of the grand canonical ensemble, the derivative that appears here is the thermal average of the fluctuations about the mean particle number. In the limit of Maxwell-Boltzmann statistics, the derivative simply reproduces the particle number density, corresponding to the fact that classical statistics have a Poisson distribution. Multiplying Eq. (A5) by k^2 and using Eq. (A5) for the electron contribution, we can write

$$k^2\epsilon(k,\omega) = k^2 + \kappa_e^2 + F_I(\omega/k), \tag{A8}$$

where we have defined the function

$$F_I(v) = -\sum_i \beta_i e_i^2 \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} \frac{\hat{\mathbf{k}}\cdot\mathbf{v}_i}{v - \hat{\mathbf{k}}\cdot\mathbf{v}_i + i\eta} f_i(\mathbf{p}_i). \tag{A9}$$

This is almost the same function F defined in Ref. [1], except that here we have handled the electron contribution separately.

Superficially it appears that F_I contains wave-vector dependence through the terms $\hat{\mathbf{k}}\cdot\mathbf{v}_i$ in the integrand of Eq. (A9); however, since we are integrating over all values of $\mathbf{v}_i$, the wave-vector direction $\hat{\mathbf{k}}$ cancels in F_I. As our notation suggests, $F_I(v)$ is indeed only a function of $v = |\mathbf{v}|$. Furthermore, because of the $i\eta$ term with $\eta > 0$ in the denominator, the function F_I is analytic in the upper complex v plane.

In evaluating the integral (8.12) in the text, we require the large-v behavior of Eq. (A9). Since the numerator of the integrand in Eq. (A9) is odd, we can expand the denominator to find the leading v behavior,

$$|v| \to \infty: \quad F_I(v) \to -\sum_i \beta_i e_i^2 \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} \frac{(\hat{\mathbf{k}}\cdot\mathbf{v}_i)^2}{v^2} f_i(\mathbf{p}_i) = -\frac{\omega_I^2}{v^2} + \mathcal{O}(v^{-4}), \tag{A10}$$

where

$$\omega_I^2 = \sum_i \frac{e_i^2 n_i}{m_i} \tag{A11}$$

is the sum of the squared ion plasma frequencies. To obtain this result, the Gaussian integral in Eq. (A10) may be calculated directly, or more elegantly, one may use Eq. (A3) to replace $(\hat{\mathbf{k}}\cdot\mathbf{v}_i)f_i$ with a derivative of f_i, after which a partial integration yields Eq. (A10).

Finally, we derive a dispersion relation that will be quite useful in evaluating Eq. (8.6) in Sec. VIII. Applying the relation

$$\mathrm{Im}\,\frac{1}{x + i\eta} = -\pi\delta(x) \tag{A12}$$

for $\eta \to 0^+$ in Eq. (A9) allows us to express the imaginary part of F_I in the form

$$\mathrm{Im}\,F_I(v) = \sum_i \beta_i e_i^2 \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} f_i(\mathbf{p}_i) v\pi\delta(v - \hat{\mathbf{k}}\cdot\mathbf{v}_i). \tag{A13}$$

From this, we can find the imaginary part of the inverse of the dielectric function:

$$\mathrm{Im}\,\frac{1}{k^2\epsilon(k,\mathbf{v}\cdot\mathbf{k})} = -\frac{\mathrm{Im}\,k^2\epsilon(k,\mathbf{v}\cdot\mathbf{k})}{|k^2\epsilon(k,\mathbf{v}\cdot\mathbf{k})|^2} = -\frac{\mathrm{Im}\,F_I(v)}{|k^2\epsilon(k,\mathbf{v}\cdot\mathbf{k})|^2} = -\frac{1}{|k^2\epsilon(k,\mathbf{v}\cdot\mathbf{k})|^2}\sum_i \beta_i e_i^2 \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} \times f_i(\mathbf{p}_i) v\pi\delta(v - \hat{\mathbf{k}}\cdot\mathbf{v}_i). \tag{A14}$$

Since the numerator in the integrand (A9) is odd, under complex conjugation we have

$$F_I(-v) = F_I(+v)^*. \tag{A15}$$

Hence using Eqs. (A8) and (A15) can write

$$\frac{\pi v}{|k^2\epsilon(k,\mathbf{v}\cdot\mathbf{k})|^2}\sum_i \beta_i e_i^2 \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} f_i(\mathbf{p}_i)\,\delta(\hat{\mathbf{k}}\cdot\mathbf{v}_i - v).$$

$$= -\frac{1}{2i}\left\{\frac{1}{k^2+\kappa_e^2+F_I(v)} - \frac{1}{k^2+\kappa_e^2+F_I(-v)}\right\}. \quad \text{(A16)}$$

APPENDIX B: SCATTERING CORRECTIONS

The electron-ion energy exchange rate computed in Sec. 12 of BPS [1] was performed under quite general conditions, with no restriction on the masses, number densities, or temperatures of the plasma components, except that the plasma be fully ionized, nondegenerate, and weakly to moderately coupled (all mild restrictions in a hot, low-Z plasma). Because of its generality, this result, which we shall present momentarily, is rather complicated. However, for most practical calculations, we can work in the high-temperature extreme-quantum limit and take advantage of the small electron-to-ion mass ratio. Under these conditions, we can use the Born approximation for the two-body scattering amplitude, and the rate coefficient collapses to the simple expression,

$$\mathcal{C}_{eI} = \frac{\omega_I^2}{2\pi}\sqrt{\frac{\beta_e m_e}{2\pi}}(\beta_e e^2 n_e)\frac{1}{2}\left[\ln\left\{\frac{8T_e^2}{\hbar^2\omega_e^2}\right\} - \gamma - 1\right], \quad \text{(B1)}$$

as previously quoted in Eq. (2.3). The purpose of this appendix is to find the subleading quantum correction to Eq. (B1).

As noted in BPS, this subleading correction is of order

$$\eta_{ei}^2 \sim 27\text{eV}/T_e, \quad \text{(B2)}$$

which, for most applications that we have in mind, is quite small. However, the leading electron degeneracy effects are of order $z_e \sim n_e a_0^3 (27\ \text{eV}/T_e)^{3/2}$, which can be comparable in size to the subleading quantum correction. Both corrections are small compared to the leading-order contribution (B1). Therefore in this appendix, we can work in the nondegenerate limit, since degeneracy effects on top of the subleading quantum effects are smaller still. Consequently, our starting point will be the nondegenerate, but otherwise rather general, expression for the rate derived in Sec. 12 of BPS. We shall simplify this rate in favor of the more realistic case of light electrons and heavy ions, exhibiting this result in Eq. (B24), an expression that is valid to all orders in the quantum parameter η_{ei}. Although this expression is clear and compact, for the purposes of this paper, however, we only need the subleading η_{ei}^2 term. This subleading correction is displayed in Eq. (B33).

The strength of the quantum effects associated with the scattering of two plasma species a and b is characterized by the dimensionless parameter

$$\bar{\eta}_{ab} = \frac{e_a e_b}{4\pi\hbar V_{ab}}, \quad \text{(B3)}$$

where the square of the thermal velocity in this expression is defined by

$$V_{ab}^2 = \frac{1}{\beta_a m_a} + \frac{1}{\beta_b m_b}. \quad \text{(B4)}$$

The extreme quantum limit, where formally $\hbar \to \infty$, is given by $\bar{\eta}_{ab} \to 0$; while the extreme classical limit, where formally $\hbar \to 0$, is given by $\bar{\eta}_{ab} \to \infty$. The former case is equivalent to the Born approximation. In Sec. 12 of BPS, the energy exchange rate from an arbitrary plasma species a to another species b,

$$\frac{d\mathcal{E}_{ab}}{dt} = -\mathcal{C}_{ab}(T_a - T_b), \quad \text{(B5)}$$

was computed to all orders in the two-body quantum-scattering parameter $\bar{\eta}_{ab}$. It was found that the rate coefficient can be written as a sum of three terms, which, in the notation of BPS, reads

$$\mathcal{C}_{ab} = \mathcal{C}_{ab,R}^{<} + (\mathcal{C}_{ab,S}^{C} + \mathcal{C}_{ab}^{\Delta Q}), \quad \text{(B6)}$$

where the last two terms have been grouped together for later convenience. These three terms are given by Eqs. (12.31), (12.25), and (12.50), respectively, in BPS:

$$\mathcal{C}_{ab,R}^{<} = \frac{\kappa_a^2\kappa_b^2}{2\pi}\left(\frac{\beta_a m_a}{2\pi}\right)^{1/2}\left(\frac{\beta_b m_b}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty} dv v^2 e^{-(1/2)(\beta_a m_a + \beta_b m_b)v^2} \times \frac{i}{2\pi}\frac{F(v)}{\rho_{\text{total}}(v)}\ln\left\{\frac{F(v)}{K^2}\right\}, \quad \text{(B7)}$$

$$\mathcal{C}_{ab,S}^{C} = -\kappa_a^2\kappa_b^2\frac{(\beta_a m_a \beta_b m_b)^{1/2}}{(\beta_a m_a + \beta_b m_b)^{3/2}}\left(\frac{1}{2\pi}\right)^{3/2} \times \left[\ln\left\{\frac{e_a e_b}{4\pi}\frac{K}{4m_{ab}V_{ab}^2}\right\} + 2\gamma\right], \quad \text{(B8)}$$

and

$$\mathcal{C}_{ab}^{\Delta Q} = -\frac{1}{2}\kappa_a^2\kappa_b^2\frac{(\beta_a m_a \beta_b m_b)^{1/2}}{(\beta_a m_a + \beta_b m_b)^{3/2}}\left(\frac{1}{2\pi}\right)^{3/2}\int_0^{\infty} d\zeta e^{-\zeta/2} \times \left[\text{Re}\ \psi\left(1 + i\frac{\bar{\eta}_{ab}}{\zeta^{1/2}}\right) - \ln\left\{\frac{\bar{\eta}_{ab}}{\zeta^{1/2}}\right\}\right]. \quad \text{(B9)}$$

Since we are using Maxwell-Boltzmann statistics throughout this section, the Debye wave number of any plasma species, including electrons, is here determined by $\kappa_b^2 = \beta_b e_b^2 n_b$. The complex-valued function $F(v)$ is defined by

$$F(v) = \int_{-\infty}^{\infty} du \frac{\rho_{\text{total}}(u)}{v - u + i\eta}, \quad \text{(B10)}$$

where $\rho_{\text{total}}(v)$ is the spectral weight,

$$\rho_{\text{total}}(v) = \sum_b \rho_b(v), \quad \text{(B11)}$$

with

$$\rho_b(v) = \kappa_b^2 \sqrt{\frac{\beta_b m_b}{2\pi}} v \exp\left\{-\frac{1}{2}\beta_b m_b v^2\right\}. \tag{B12}$$

This is similar to the function F_I introduced in the previous appendix, except here the sum extends over all plasma species, including electrons. Note the dependence in Eqs. (B7) and (B8) on an unspecified parameter K, with the only restriction being that K has units of a wave number. This is an artifact of the calculational procedure, and it was shown in BPS that the total rate $\mathcal{C}_{ab}$ is indeed independenet of K, as this parameter cancels in the sum between Eqs. (B7) and (B8). As a matter of technical convenience, we will henceforth set $K=\kappa_e$ throughout the rest of this appendix [the simplified result (B18) only holds under this condition]. Finally, we should note that the reduced mass is determined by $1/m_{ab}=1/m_a+1/m_b$, and

$$\psi(z) = \frac{1}{\Gamma(z)}\frac{d\Gamma(z)}{dz} \tag{B13}$$

is the logarithmic derivative of the gamma function.

Specializing to electrons and ions (in which $a=e$ and $b=i$), we can employ Eq. (B12) to write Eq. (B7) in the form

$$\begin{aligned}\mathcal{C}_{eI,R}^{<} &\equiv \sum_i \mathcal{C}_{ei,R}^{<} \\ &= \frac{\kappa_e^2}{2\pi}\left(\frac{\beta_e m_e}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty} dv v e^{-(1/2)\beta_e m_e v^2}\frac{i}{2\pi} \\ &\quad\times\frac{1}{\rho_{\text{total}}(v)}\sum_i \rho_i(v)F(v)\ln\left\{\frac{F(v)}{\kappa_e^2}\right\}.\end{aligned} \tag{B14}$$

This expression greatly simplifies since the electron is so much lighter than the ions. In virtually all practical applications, the electron and ion temperatures are never excessively disparate, and we can therefore impose the mild restriction

$$\beta_e m_e \ll \beta_i m_i. \tag{B15}$$

We will refer to the condition (B15) as the $m_e\to 0$ limit, and the ratio $\beta_e m_e/\beta_i m_i$ can then be used as a small dimensionless expansion parameter. For example, to leading order in this parameter we find

$$\frac{1}{\rho_{\text{total}}(v)}\sum_i \rho_i(v) = 1 + \mathcal{O}\left(\frac{\beta_e m_e}{\beta_i m_i}\right)^{1/2}, \tag{B16}$$

which allows us to express Eq. (B14) as

$$\begin{aligned}\mathcal{C}_{eI,R}^{<} &= \frac{\kappa_e^2}{2\pi}\left(\frac{\beta_e m_e}{2\pi}\right)^{1/2}\frac{i}{2\pi}\int_{-\infty}^{\infty} dv v F(v)\ln\left\{\frac{F(v)}{\kappa_e^2}\right\} \\ &\quad\times\left[1+\mathcal{O}\left(\frac{\beta_e m_e}{\beta_i m_i}\right)^{1/2}\right].\end{aligned} \tag{B17}$$

We have omitted the exponential in the integrand of Eq. (B14), since the function $F(v)$ provides enough convergence at large values of v to allow the $m_e\to 0$ limit to be brought inside the integral. The analytic properties of $F(v)$ allow us to perform the v integral in Eq. (B17) using contour integral techniques, in much the same manner as we did in the discussion following Eq. (8.9) in the text. The result is Eq. (12.44) of BPS, which reads

$$\beta_e m_e \ll \beta_i m_i: \quad \mathcal{C}_{eI,R}^{<} = -\frac{1}{2}\frac{\kappa_e^2}{2\pi}\left(\frac{\beta_e m_e}{2\pi}\right)^{1/2}\sum_i \omega_i^2, \tag{B18}$$

a much simpler expression indeed. For the electrons and ions we are considering, we can also drop the term of order $\beta_e m_e/\beta_i m_i$ in Eq. (B3), allowing us to express the quantum parameter of Eq. (B4) as

$$\bar{\eta}_{ei} = \frac{Z_i e^2}{4\pi\hbar}\sqrt{\frac{m_e}{T_e}} = Z_i\bar{\eta}_e. \tag{B19}$$

On occasion, we will express the quantum parameter in terms of the binding energy of the hydrogen atom,

$$\epsilon_H = \frac{1}{2}\left(\frac{e^2}{4\pi}\right)^2\frac{m_e}{\hbar^2} \simeq 13.606\ \text{eV}, \tag{B20}$$

so that

$$\bar{\eta}_{ei}^2 = Z_i^2\frac{2\epsilon_H}{T_e}. \tag{B21}$$

We see that $\bar{\eta}_{ei}\ll 1$ when T_e reaches the keV scale, illustrating that quantum corrections are important at high temperatures. Finally, we can drop terms of order $\beta_e m_e/\beta_i m_i$ in the leading coefficients of Eqs. (B8) and (B9), thereby allowing us to write

$\beta_e m_e \ll \beta_i m_i$:

$$\begin{aligned}\mathcal{C}_{eI,S}^{C} + \mathcal{C}_{eI}^{\Delta Q} &= -\frac{\kappa_e^2}{2\pi}\left(\frac{\beta_e m_e}{2\pi}\right)^{1/2}\sum_i \omega_i^2\left(\ln\left\{\frac{Z_i e^2}{4\pi}\frac{\kappa_e}{4T_e}\right\}\right. \\ &\quad + 2\gamma + \frac{1}{2}\int_0^{\infty} d\zeta e^{-\zeta/2}\left[\text{Re}\,\psi\left(1+i\frac{Z_i\bar{\eta}_e}{\zeta^{1/2}}\right)\right. \\ &\quad \left.\left.- \ln\left\{\frac{Z_i\bar{\eta}_e}{\zeta^{1/2}}\right\}\right]\right).\end{aligned} \tag{B22}$$

Unlike Eq. (B18), which only holds for the sum over ion components, the result (B22) actually holds component by component. Performing the ζ integral for the last term in Eq. (B22) gives

$\beta_e m_e \ll \beta_i m_i$:

$$\begin{aligned}\mathcal{C}_{eI,S}^{C} + \mathcal{C}_{eI}^{\Delta Q} &= \frac{\kappa_e^2}{2\pi}\left(\frac{\beta_e m_e}{2\pi}\right)^{1/2}\sum_i \frac{\omega_i^2}{2}\left[\ln\left\{\frac{8m_e T_e}{\hbar^2\kappa_e^2}\right\}\right. \\ &\quad \left. - 3\gamma - \int_0^{\infty} d\zeta e^{-\zeta/2}\,\text{Re}\,\psi\left(1+i\frac{\bar{\eta}_{ei}}{\zeta^{1/2}}\right)\right].\end{aligned} \tag{B23}$$

The rate $\mathcal{C}_{eI}$ is given by adding Eqs. (B18) and (B23), which can be written as

$$\beta_e m_e \ll \beta_i m_i: \quad \mathcal{C}_{eI} = \frac{\kappa_e^2}{2\pi}\left(\frac{\beta_e m_e}{2\pi}\right)^{1/2}\frac{1}{2}\sum_i \omega_i^2\left[\ln\left\{\frac{8m_e T_e}{\hbar^2\kappa_e^2}\right\} - \gamma - 1 - \Delta_i(\bar{\eta}_{ei})\right], \tag{B24}$$

with

$$\Delta_i(\bar{\eta}_{ei}) = \int_0^\infty d\zeta e^{-\zeta/2}\left[\mathrm{Re}\ \psi\left(1 + i\frac{\bar{\eta}_{ei}}{\zeta^{1/2}}\right) + \gamma\right]. \tag{B25}$$

This expression is accurate to leading and next-to-leading order in the plasma coupling, and to all orders in $\bar{\eta}_{ei}$, with no restriction on the temperature [apart from requiring the mild constraint (B15) and that the plasma coupling be small].

With the aid of Eq. (10.17) of BPS,

$$\mathrm{Re}\ \psi(1 + i\bar{\eta}\zeta^{-1/2}) + \gamma = \sum_{k=1}^{\infty}\frac{1}{k}\frac{\bar{\eta}^2}{k^2\zeta + \bar{\eta}^2}, \tag{B26}$$

and writing

$$\frac{1}{\zeta + \bar{\eta}^2/k^2} = \frac{d}{d\zeta}\ln\left\{\zeta + \frac{\bar{\eta}^2}{k^2}\right\}, \tag{B27}$$

a partial integration now gives

$$\Delta_i(\bar{\eta}_{ei}) = \bar{\eta}_{ei}^2\sum_{k=1}^{\infty}\frac{1}{k^3}\left[\ln\left\{\frac{k^2}{\bar{\eta}_{ei}^2}\right\} + \frac{1}{2}\int_0^\infty d\zeta e^{-\zeta/2}\ln\left\{\zeta + \frac{\bar{\eta}_{ei}^2}{k^2}\right\}\right]. \tag{B28}$$

In the latter form, we can easily extract the leading order term $\bar{\eta}_{ei}$, since we can use the limit

$$\bar{\eta}_{ei} \to 0: \quad \frac{1}{2}\int_0^\infty d\zeta e^{-\zeta/2}\ln\left\{\zeta + \frac{\bar{\eta}_{ei}^2}{k^2}\right\} \to \int_0^\infty d(\zeta/2)e^{-\zeta/2}(\ln\{\zeta/2\} + \ln 2) = -\gamma + \ln 2. \tag{B29}$$

Using now

$$\zeta(3) = \sum_{k=1}^{\infty}\frac{1}{k^3} = 1.202\,05\ldots, \tag{B30}$$

and

$$\zeta'(3) = -\sum_{k=1}^{\infty}\frac{1}{k^3}\ln k = -0.198\,12\ldots, \tag{B31}$$

we can express the subleading quantum correction as

$$\Delta_i(\bar{\eta}_{ei}) \simeq \bar{\eta}_{ei}^2\left\{\zeta(3)\left[\ln\left\{\frac{2}{\bar{\eta}_{ei}^2}\right\} - \gamma\right] - 2\zeta'(3)\right\}. \tag{B32}$$

Using Eq. (B21), we can express the rate (B24) to leading and next-to-leading order in $\bar{\eta}_{ei}$ as

$$\mathcal{C}_{eI} = \frac{\omega_I^2}{2\pi}\sqrt{\frac{\beta_e m_e}{2\pi}}\kappa_e^2\frac{1}{2}\left[\ln\left\{\frac{8T_e^2}{\hbar^2\omega_e^2}\right\} - \gamma - 1\right] - \frac{1}{2\pi}\sqrt{\frac{\beta_e m_e}{2\pi}}\kappa_e^2\frac{\epsilon_H}{T_e}\sum_i Z_i^2\omega_i^2\left[\zeta(3)\left(\ln\left\{\frac{T_e}{Z_i^2\epsilon_H}\right\} - \gamma\right) - 2\zeta'(3)\right]. \tag{B33}$$

The correction is of order ϵ_H/T_e, which can be of comparable size to the leading degeneracy correction.

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[16] The validity of using the Born approximation for the scattering amplitude is discussed in more detail in Ref. [1], particularly in the work leading to its Eq. (12.54).

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